ESSAYS ON UNSECURED CREDIT, UNCERTAINTY, AND LEARNING

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ESSAYS ON UNSECURED CREDIT, UNCERTAINTY, 
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ABSTRACT

If lending contracts in an economy take the form of unsecured, non-state-contingent debt, a recession will often be associated with an increase in defaults and a reduction in the supply of credit, which amplifies the contraction. Within this context, I consider the case when the length of an unfolding recession is not immediately obvious; instead, it takes people time to learn about a persistent recession. I apply this scenario to models where the unsecured credit is represented by mortgage loans extended to households and by sovereign debt extended to an emerging economy. I find that in both cases, accounting for uncertainty and learning has a potential to improve the empirical performance of the models.

Chapter 1 explores the U.S. housing market where house prices show a lot of inertia. I develop a general-equilibrium model with the market for housing and mortgages and introduce uncertainty regarding the persistence of business cycles. I show that uncertainty allows the model to better account for the sluggish dynamics of the housing market. In Chapter 2, I use key U.S. macroeconomic data to empirically estimate the structural model developed in Chapter 1. In order to compare the performance of the models with and without uncertainty, I use likelihood-based estimation methods. The model with uncertainty proves to be better capable of mimicking the long-lasting changes in house prices and other observable variables.

Chapter 3 contains a theoretical model of a small open emerging economy that looks to refinance its sovereign debt during an unfolding recession of uncertain length. A long
recession implies a higher chance of default in the future; a short recession means quick recovery and solvency. Uncertainty about the unfolding scenario adds price risk to long-term bonds and makes them costly to the borrower. Investors’ preferences shift towards short-term bonds which mature before a lot of the uncertainty is resolved and before credit events are likely to happen. Such uncertainty helps explain the empirical fact that emerging economies tend to borrow short term during economic downturns.
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Chapter 1

LEARNING AND THE MARKET FOR HOUSING

1.1 Introduction

The financial crisis of 2007–2009 has identified the elephant in the room, which is the sheer size of the housing market and its influence on the aggregate economy. Before the crisis, the housing market had been booming for over a decade; rising house prices fueled the lenders’ desire to create new, risky types of mortgages and offer them to the widest set of households, including those with questionable credit history and unreported income. By end of 2006, U.S. households were highly leveraged, with 10 trillion dollars worth of debt, 70% of which was due to mortgages. The end of the housing boom forced millions of mortgages into negative home equity, leading to a surge in mortgage foreclosures, massive mortgage write-offs by banks and a collapse of the multi-trillion-dollar market for mortgage-backed securities. The banking sector experienced losses in net worth and a credit crunch. What unfolded was the most severe financial crisis and the longest recession in decades. Events like these require that economic theory provides a better understanding of the housing market.

What is particularly puzzling about the housing market is how slow it is to adjust (see figure 1.1). Following the crisis, the house price index continued to decline steadily for five years; and the mortgage foreclosure rate has hovered above the one-percent mark for over three years; more than double its average value for 2002–2005. However, it is common knowledge that market prices should quickly absorb all the information about the current and future state of economy. For example, the S&P500 Index shows that the downward price adjustment in the market for capital lasted for five quarters. To explain this feature

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1Quarterly Report on Household Debt and Credit, Federal Reserve; Mian and Sufi (2009) evaluate the contribution of household leverage to severity of the 2007–2009 recession

2There was a brief increase in house prices following 2008, when the Federal Reserve announced its large-scale MBS purchase program. The effect of the program on house prices is unclear, though; see Fuster and Willen (2010) for a discussion of the program’s effect on house price.
of housing market dynamics, I suggest that economic agents did not initially recognize the scope and length of the Great Recession. They observed the deteriorating economy, but did not expect the decline to be persistent. In effect, they were over-optimistic. Households were betting on temporary recession and on a continued housing boom; they were willing to keep purchasing houses and obtaining large mortgages. As the economic downturn continued, the agents eventually realized its scope. Such gradual recognition of a persistent recession can explain the slow reaction of house prices. This mechanism can potentially account for the slow evolution of other variables as well. For example, an unexpected decline in house prices is the major driver of mortgage foreclosures.\footnote{See Gerardi et al. (2008)}

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\footnote{See Gerardi et al. (2008)}
The goal of this paper is to study uncertainty about the economic growth as an explanation for sluggish dynamics in the market for housing. I take three steps towards this goal. First, I build a dynamic stochastic general equilibrium model with an endogenous market for housing and mortgages that is driven by both transitory and persistent shocks. There is a rich structure of shocks in the model, but I only allow the technology processes to have shocks with different persistence. Technology growth would have two components: a transitory component, or white noise; and a persistent component, an AR(1) process. Second, I consider the situation of imperfect knowledge when economic agents cannot immediately recognize the persistence of shocks. They can only observe the aggregate technology growth, but cannot observe its individual components. Using Kalman updating, agents gradually learn about each component’s contribution by observing the evolution of growth through time. And third, I evaluate the ability of the model with such uncertainty to better explain the observed sluggishness in housing-market data. To that end, I look at dynamic features of the model, such as impulse-response functions. I conduct a more thorough likelihood-based empirical estimation for the models with perfect and imperfect knowledge, compare their performance, and present the results in Chapter 2.

There are several arguments to motivate the assumption of imperfect knowledge about the persistence of changes in technology growth. First, it is hard for a model with rational expectations and perfect knowledge to generate any endogenous response of prices to a shock other than a sharp swing followed by gradual recovery back to the steady state. Learning under imperfect knowledge is a powerful tool that may protract the dynamic response of the model variables, as well as improve their co-movement.4

Second, there is empirical evidence to support the assumption of imperfect knowledge and learning, especially about the rate of technological growth. Edge et al. (2007) study the long-run TFP growth rates predicted by professional forecasters5 and find that, despite a plethora of complicated tools used by the professionals, their predictions are strikingly

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4Edge et al. (2007) and Huang et al. (2009) provide similar arguments.
5Survey of Professional Forecasters, projections by the Congressional Budget Office.
close to a simple learning mechanism based on linear Kalman updating. In addition, Foote et al. (2012) argue that during the Great Recession, mortgage market participants did not know the state of economy; they had beliefs that were *ex-post* over-optimistic and acted rationally subject to them.

Finally, the assumption of imperfect knowledge about the shocks can help explain why the price adjusts slower in the housing market than in the market for capital. Capital market participants are financial market professionals who closely monitor the state of the economy. The housing market encompasses virtually every household. It is reasonable to believe that participants in the housing market have less knowledge about the state of economy and the future of economic growth, so that they have to rely on learning more.

The novelty of my work is that it combines (i) a market for housing and mortgages with (ii) imperfect knowledge and learning into a model that is (iii) tractable and can be tested directly against the data. The crisis of 2007–2009 has brought the housing market to many economists’ attention, and has motivated the development of models of housing and collateralized household debt. Examples include Chatterjee and Eyigungor (2011), Corbae and Quintin (2013), Garriga and Schlangenau (2009), Iacoviello and Pavan (2012), Monacelli (2009), etc. The ability to track house prices, mortgage default rates, loan-to-value ratios, and mortgage risk premiums endogenously in a general equilibrium set-up is a recent achievement of this field that has become possible thanks to an increased interest in the topic. To build a tractable model, I apply the structure of idiosyncratic investment risk developed by Bernanke et al. (1999) to the market for mortgages. In this respect, my work is similar to Forlati and Lambertini (2011). Neri and Iacoviello (2008) study the impact of the housing market on aggregate economy. They introduce a rich technology structure that accounts for long-run growth and a portion of short-run fluctuations in house price and residential investment. My model has a similar structure of technology but expands it to incorporate persistent and transitory shocks, as well as imperfect knowledge about

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6The authors develop a dynamic new-Keynesian model to study the shocks originating from the market for mortgages and their impact on the aggregate economy.
them.

The literature on imperfect knowledge and learning has also caught up after the crisis. I employ the learning mechanics similar to Gilchrist and Saito (2006). The authors extend the dynamic new-Keynesian model with financial frictions (Bernanke et al., 1999) to implement imperfect knowledge about the persistence of shocks and study its implications for monetary policy. Orphanides and Williams (2007) study the implications of perpetual learning (a rule of thumb based on simple OLS to forecast future prices) for the optimal monetary policy. Eusepi and Preston (2008) show that perpetual learning helps an RBC model amplify and propagate investment and labor supply. Fuster et al. (2010) entertain the finding that agents rely heavily on the recent observations to form their forecasts and show that such assumption helps account for volatility of asset prices and cyclical properties of equity returns in an asset-pricing model.

Given the advances in the literature on housing and the capability of the literature on learning to improve, or, at least, affect model dynamics, it is odd that little has been done to combine the two. One known example is Burnside et al. (2011) who develop a model with heterogeneous beliefs and an intricate learning mechanism. The authors argue that it is difficult to generate protracted house price dynamics in case of homogeneous beliefs because a change in beliefs quickly translates into changes in prices. Their learning mechanism gradually spreads the beliefs like an infection, creating protracted dynamics in house price. I view my work in this respect as an attempt to account for protracted dynamics in house price in a model with homogeneous beliefs, and in a simpler general-equilibrium set-up. In my model, imperfect knowledge and learning make changes in homogeneous beliefs gradual and create sluggishness.

The paper proceeds as follows: section 1.2 defines the model and discusses its key assumptions; section 1.3 describes the strategy to evaluate the model empirically; section 1.4 shows the results of empirical estimation; and section 1.5 concludes.
1.2 Model

I design a dynamic stochastic general equilibrium model with endogenous market for mortgages. Production consists of two sectors: consumption good production and housing construction. Households derive utility from both consumption good and housing stock. There are two groups of households: impatient borrowers and patient savers. Borrowers supply fixed labor and earn wage; purchase consumption good and housing stock; and find it optimal to use housing stock as collateral to borrow mortgages. Savers purchase consumption good and housing stock; they find it optimal to lend mortgages to the borrowers and invest into capital in both production sectors and earn mortgage interest and capital rent. Savers also supply fixed labor and earn wage. Since the savers originate the capital, they claim all the profit from production if such exists. Time is discreet; one period equals one quarter. The description of the model is more efficient if I explain the workings of the mortgage market first.

1.2.1 Mortgage contracts

To design a tractable market for mortgages with endogenous default rate, loan-to-value ratio, and mortgage premium, I apply the mechanics of endogenous borrowing constraint for entrepreneurs of Bernanke et al. (1999) to mortgages.\(^7\)

For tractability, I make three simplifying assumptions. First, there are only one-period mortgages available. In reality, mortgages usually have 30-year terms and households at different stages of mortgage amortization behave differently; it is sufficient to mention that households with recently acquired mortgages have a larger outstanding debt compared to the value of the house and are more likely to go underwater and default.\(^8\) However, tracking the distribution of households across the stages of mortgage loan repayment would overcomplicate the model. I choose to focus on aggregate behavior of key mortgage market indicators and expect that the model with one-period contracts does not change the results.

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\(^7\)This approach is introduced by Forlati and Lambertini (2011).

\(^8\)See Garriga and Schlagenhauf (2009) for an example.
qualitatively. Second, the model features fixed-rate mortgages only: interest rate is known at the time of loan origination. There are adjustable-rate mortgages and contracts with hybrid interest schedules available in the market, but the vast majority of mortgages are FRM loans. Moreover, for a model with one-period loans, the choice between ARM and FRM contracts is not important, since mortgage interest rates are re-negotiated every period. Finally, the third simplification is that there is no recourse or punishment in case of default. It implies that the household chooses to default on mortgage whenever the value of the house is less than the outstanding debt.

Consider a household agent $i$ that purchases a house $H_{i,t}$ in period $t$ and pays $H_{i,t}P_t$ for it, where $P_t$ is the house price in terms of consumption good. To finance this purchase, the agent can use the house as collateral and obtain a one-period loan $B_{i,t}$ with a fixed interest rate $\bar{r}_{m,i,t}$. Next period, the outstanding debt is $B_{i,t}(1 + \bar{r}_{m,i,t})$, and the value of the house is $H_{i,t}P_{t+1}\Omega_{t+1}(1 - \delta_h)\omega_{i,t+1}$, where $\delta_h$ is the housing stock depreciation rate and $\Omega_{t+1}$ is the aggregate shock to the housing stock size. The term $\omega_{i,t+1}$ is the idiosyncratic shock to agent $i$’s housing stock size that has a log-normal distribution centered at 1:

$$\omega_{i,t} \sim F(\omega) \text{ i.i.d.}, \text{ such that } \ln \omega_{i,t} \sim N\left(-\frac{1}{2}\sigma^2_\omega, \sigma^2_\omega\right) \Rightarrow E[\omega_{i,t}] = 1.$$ 

In absence of recourse or punishment, the agent will repay the loan if the value of the house exceeds the outstanding debt:

$$H_{i,t}P_{t+1}\Omega_{t+1}(1 - \delta_h)\omega_{i,t+1} \geq B_{i,t}(1 + \bar{r}_{m,i,t}).$$

For convenience, define the threshold value of idiosyncratic housing stock shock:

$$\bar{\omega}_{i,t+1} = \frac{B_{i,t}(1 + \bar{r}_{m,i,t})}{H_{i,t}P_{t+1}(1 - \delta_h)\Omega_{t+1}}. \quad (1.1)$$

---

9Corbae and Quintin (2013) study the contribution of non-traditional mortgages to the recession.

10FHFA’s Monthly Interest Rate Survey reports that in 1990–2010, the share of FRM mortgages averaged around 70%.
The household agent will repay the debt if the shock realization is above the threshold next period: \( \omega_{i,t+1} \geq \bar{\omega}_{i,t+1} \).

The most commonly discussed complication of a model with endogenous default is that only a fraction of households default on their loans in equilibrium. To account for it, the model needs heterogeneous households and, as a result, one must explicitly track the endogenous distribution of households in order to compute the default rate and aggregate prices in the economy. This would tremendously complicate the solution of a general equilibrium model, and a convenient approach to this problem is to assume perfect risk-sharing within each household. Let each household be a unit mass of household agents, where each agent \( i \) conducts the policy that is optimal for the aggregate household. Then, if a variable \( Z_t \) is a part of the household’s optimal policy, it is true that

\[
Z_t = \int_0^1 Z_{i,t} \, di, \quad \text{and} \quad Z_{i,t} = Z_t.
\]

Thus, within each household, agents purchase equally-sized houses and get equivalent mortgage contracts, so the same threshold \( \bar{\omega}_{t+1} \) is applicable to every agent. Each agent is subject to agent-specific idiosyncratic shock \( \omega_{i,t+1} \), and a fraction of agents will default, but the household pools the ex-post payoffs from mortgage arrangements and is not subject to idiosyncratic risk. Such set-up renders the model unable to describe potentially interesting effects of mortgage market on household wealth distribution; however, it keeps the solution highly tractable and still retains the essential interplay between the housing purchases, chance of default, and the borrowing constraint. Henceforth, ‘household’ stands for a continuum of household agents, and index \( i \) is dropped.

The mortgage contract is an arrangement between the saver and the borrower. At period \( t \), the borrower purchases \( H_t \) and gets a loan \( B_t \) at fixed rate \( \bar{r}_{m,t} \), so that next period, her payoff will be

\[
H_t P_{t+1} (1 - \delta_h) \Omega_{t+1} \int_{\bar{\omega}_{t+1}}^{\infty} \omega \, dF(\omega) - B_t (1 + \bar{r}_{m,t}) \int_{\bar{\omega}_{t+1}}^{\infty} dF(\omega).
\]
That is, she only retains the houses and repay the loans of the agents who do not default. In case of a default, the loan is repudiated and the house is lost to the saver, so the payoff is zero. The saver provides $B_t$ at period $t$, so next period, she will collect

$$H_t P_{t+1} \Omega_{t+1} (1 - \delta_h) (1 - \mu) \int_{\bar{\omega}_{t+1}}^{\omega} \omega dF(\omega) + B_t (1 + \bar{r}_{m,t}) \int_{\bar{\omega}_{t+1}}^{\infty} dF(\omega).$$

Fraction $\mu$ captures the cost of default paid by the lender. Foreclosed property is usually sold at a significant discount; there may be legal fees, debt collector’s commission, etc. For such costs, it seems appropriate to assume that they are proportionate to the size of the house.

For brevity of notation, define

$$\Gamma(\bar{\omega}_t) = \int_{0}^{\bar{\omega}_t} \omega dF(\omega) + \bar{\omega}_t \int_{\bar{\omega}_t}^{\infty} dF(\omega), \quad (1.2)$$

$$G(\bar{\omega}_t) = \int_{0}^{\bar{\omega}_t} \omega dF(\omega), \quad (1.3)$$

where $\Gamma(\bar{\omega}_t)$ is the debt repaid to the mortgage lender expressed as the share of the housing stock collateral, and $G(\bar{\omega}_t)$ is the average idiosyncratic shock to housing stock associated with repudiated mortgages. Using equations (1.1)–(1.3) and the fact that $E[\omega] = 1$, the borrower’s payoff becomes

$$H_t P_{t+1} (1 - \delta_h) \Omega_{t+1} (1 - \Gamma(\bar{\omega}_{t+1})), \quad (1.4)$$

and the saver’s payoff is

$$H_t P_{t+1} (1 - \delta_h) \Omega_{t+1} (\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})).$$

In effect, a mortgage contract involves two parties co-paying for a house and splitting the value of the house between them upon the mortgage contract settlement: the saver claims $\Gamma(\cdot)$, the borrower retains $1 - \Gamma(\cdot)$, and $\mu G(\cdot)$ is lost due to default. To further the intuition,
let \( r_{m,t+1} \) be the realized saver’s return on mortgage:

\[
B_t(1 + r_{m,t+1}) = H_t P_{t+1}(1 - \delta_h) \Omega_{t+1} (\Gamma(\tilde{\omega}_{t+1}) - \mu G(\tilde{\omega}_{t+1}))
\]  

(1.5)

Using (1.5), the borrower’s payoff can be intuitively rewritten as

\[
H_t P_{t+1}(1 - \delta_h) \Omega_{t+1} (1 - \mu G(\omega_{t+1})) - B_t(1 + r_{m,t+1})
\]  

(1.6)

So, the mortgage arrangement is effectively such that the saver earns the ex-post return \( r_{m,t+1} \) on the loan; while the borrower retains the house, repays the debt at ex-post rate \( r_{m,t+1} \), and ends up bearing the cost of default.

1.2.2 Households

The total population is fixed at 1; a fraction \( \Psi \) are impatient and the remaining \( 1 - \Psi \) are patient households. The patient household’s discount factor is higher than that of the impatient one: \( 1 > \hat{\beta} > \beta > 0 \). (Note that variables and parameters marked with a hat are pertinent to savers; the ones with no accent—to borrowers.) As a result, in equilibrium, the impatient households are borrowers and the patient households are savers.\(^\text{11}\) Savers provide borrowers with mortgages and invest into capital; they own the firms in the economy.

The infinitely-lived households derive utility from consumption good and housing services. The lifetime utility function is time-separable and logarithmic:

\[
U_t = \sum_{j=0}^{\infty} \beta^j E_t [U(C_{t+j}, H_{t+j})], \quad \text{where} \quad U(C_t, H_t) = \nu_t (\ln C_t + \psi_t \ln H_t);
\]

where \( \psi_t \) is the weight of housing in the utility and \( \nu_t \) is the inter-temporal preference variable. Both \( \psi_t \) and \( \nu_t \) are exogenous shock processes. A positive innovation to \( \psi_t \) corresponds to an increase in housing demand; and a positive innovation to \( \nu_t \) makes

\(^{11}\)Using the households’ constrained optimization problems, it is straightforward to prove that the difference in discount factors guarantees that the borrowers find it optimal to borrow and not save, while savers find it optimal to save and not borrow.
households less thrifty, since they value current consumption and housing more compared to their future values, given that the increase in $\nu_t$ is temporary.

Each period, savers maximize the expected utility by choosing the levels of consumption $\hat{C}_t$, housing stock $\hat{H}_t$, mortgage lending $\hat{S}_t$, and purchases of capital in consumption sector $\hat{K}_{y,t}$ and construction sector $\hat{K}_{x,t}$, subject to the budget constraint:

$$\hat{C}_t + \hat{H}_t P_t + \hat{S}_t + \frac{\hat{K}_{y,t}}{A_k,t} + \hat{K}_{x,t} = \hat{H}_{t-1} P_t (1 - \delta_h) \Omega_t + (1 + r_{m,t}) \hat{S}_{t-1} +$$

$$+ (r_{x,t} + 1 - \delta_x) \hat{K}_{x,t-1} + \left( R_{y,t} + \frac{1 - \delta_y}{A_k,t} \right) \hat{K}_{y,t-1} + W_t + \hat{\Pi}_t. \tag{1.7}$$

$A_{k,t}$ is the shock to the cost of consumption-sector capital measured in units of consumption good; this technology process mostly refers to non-tangible capital and is more applicable to consumption good production rather than construction.\(^{12}\) Savers’ wealth includes housing stock retained from the previous period; return on mortgage lending; return on capital investment in both sectors equal to capital rent plus the retained capital stock net of depreciation; wage $W_t$; and construction-sector profit $\hat{\Pi}_t$. As I explain in section 1.2.3, only the construction sector earns non-zero profit.

The optimality conditions are the following:

$$U'_{\hat{C},t} P_t = U'_{\hat{H},t} + \hat{\beta} E_t \left[ U'_{\hat{C},t+1} (1 - \delta_h) P_{t+1} \Omega_{t+1} \right] \tag{1.8}$$

$$U'_{\hat{C},t} = \hat{\beta} E_t \left[ U'_{\hat{C},t+1} \left( R_{y,t+1} + \frac{1 - \delta_y}{A_k,t+1} \right) \right] \tag{1.9}$$

$$U'_{\hat{C},t} = \hat{\beta} E_t \left[ U'_{\hat{C},t+1} (r_{x,t+1} + 1 - \delta_x) \right] \tag{1.10}$$

$$U'_{\hat{C},t} = \hat{\beta} E_t \left[ U'_{\hat{C},t+1} (r_{m,t+1} + 1) \right] \tag{1.11}$$

Equation (1.8) is the first-order condition with respect to housing stock: on the left hand, an increment in housing stock would cost some utility due to sacrificed consumption, but on the right hand, the household gains utility because housing stock directly increases utility

\(^{12}\)Neri and Iacoviello (2008) consider this process as well. A more intuitive formulation is $\hat{K}_{y,t} = A_{k,t} \hat{C}_{y,t}$, where $\hat{C}_{y,t}$ is the amount of consumption good spent on consumption-sector capital.
and because it adds to the next period’s wealth. Euler equations (1.9)–(1.11) predict that mortgage lending and capital investment should all yield the same expected return.

The borrowers choose consumption \( C_t \), housing purchase \( H_t \), and the mortgage contract \( \{B_t, \bar{r}_{m,t}\} \) to maximize the expected utility subject to the constraints:

\[
C_t + H_t P_t - B_t = W_t + H_{t-1} P_t (1 - \delta_h) \Omega_t (1 - \Gamma(\omega_t)) \tag{1.12}
\]

\[
B_t = E_t \left[ H_t P_{t+1} (1 - \delta_h) \Omega_{t+1} (\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})) / (1 + r_{m,t+1}) \right] \tag{1.13}
\]

\[
\bar{\omega}_t = \frac{B_{t-1} (1 + \bar{r}_{m,t-1})}{H_{t-1} P_t (1 - \delta_h) \Omega_t} \tag{1.14}
\]

According to the budget constraint (1.12), the borrower funds her purchases by wage income \( W_t \) and mortgage payoffs. The saver’s participation constraint (1.13) and the definition of the default threshold (1.14) constrain the borrower’s maximization problem: assumably, the borrower knows what implications a mortgage contract \( \{B_t, \bar{r}_{m,t}\} \) has for the chances of default and that her mortgage repayment must provide the expected return \( 1 + r_{m,t+1} \) required by the saver.

The optimality conditions are the following:

\[
U'_{C,t} P_t = U'_{H,t} + \beta E_t \left[ U'_{C,t+1} P_{t+1} (1 - \delta_h) \Omega_{t+1} (1 - \Gamma(\omega_{t+1})) \right] \tag{1.15}
\]

\[
+ U'_{C,t} E_t \left[ P_{t+1} (1 - \delta_h) \Omega_{t+1} (\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})) / (1 + r_{m,t+1}) \right]
\]

\[
\beta E_t \left[ U'_{C,t+1} \Gamma'(\bar{\omega}_{t+1}) \right] = U'_{C,t} E_t \left[ (\Gamma'(\bar{\omega}_{t+1}) - \mu G'(\bar{\omega}_{t+1})) / (1 + r_{m,t+1}) \right] \tag{1.16}
\]

The first-order condition with respect to housing stock (1.15) is similar to that of the saver, except that, apart from direct impact on utility and an increase in next period’s wealth, one more benefit of housing stock for the borrower is that it serves as collateral and increases access to debt (the last term on the right-hand side). Equation (1.16) is the first-order condition with respect to mortgage interest rate \( \bar{r}_{m,t} \). Recall that the share of the housing stock claimed by the saver due to the mortgage contract is \( \Gamma(\bar{\omega}) - \mu G(\bar{\omega}) \); it is an increasing function of \( \bar{\omega} \) (see section A.1.1 of the Appendix). That is, \textit{ceteris paribus,
a higher mortgage rate $\tilde{r}_{m,t}$ increases the chance of default and decreases the borrower’s payoff from mortgage; but, on the other side, the saver will claim a larger fraction of the housing stock, which, in effect, expands the borrower’s capability to get a larger debt.

### 1.2.3 Production

Consumption good sector employs labor $n_{y,t}$ and capital stock $K_{y,t-1}$\(^{13}\) to produce the consumption good $Y_{y,t}$. There is a technology shock $A_{y,t}$ that is specific to consumption good production. The resulting output is $Y_{y,t} = (A_{y,t}n_{y,t})^{1-\alpha_y}K_{y,t-1}^{\alpha_y}$. The sector is competitive and the firms earn zero profit. Profit-maximization gives the standard expressions for wage and capital rent in the sector:

\[
W_{y,t} = (1 - \alpha_y)A_{y,t}^{1-\alpha_y}n_{y,t}^{-\alpha_y}K_{y,t-1}^{\alpha_y}, \quad (1.17)
\]
\[
R_{y,t} = \alpha_y A_{y,t}^{1-\alpha_y}n_{y,t}^{-1}K_{y,t-1}^{-\alpha_y}, \quad (1.18)
\]

The consumption good has three uses: it can be consumed by households, used to purchase capital, and an amount $X_t$ of consumption good can be used as an intermediary input for housing construction: $Y_{y,t} = AC_t + IK_t + X_t$.

Construction sector employs labor $n_{x,t}$, capital stock $K_{x,t-1}$, consumption good $X_t$, and aggregate housing stock retained from the previous period $\bar{H}_t$ to produce new housing stock. Housing construction includes installation of household appliances and furnishing, which motivates the consumption good being a part of housing production function. It also simplifies the derivation of house price and construction sector’s capital rent and wage. As for the retained housing stock, there are two good reasons to make it one of the construction sector’s inputs. First, housing construction, or, more broadly, creation of additional housing stock, includes renovations of the existing houses. And second, inclusion of housing stock in housing construction function adds inertia to it. For an example, a higher level of housing

---

\(^{13}\)I use the time index to indicate the period at which the value of the variable is known. In terms of timing, consumption good production happens at the beginning of the period, and savers have decided upon the scale of capital investment at the end of the previous period.
construction today would add to the retained housing stock tomorrow and, hence, add to housing construction tomorrow as well, and so on. Effectively, it adds volatility to house prices, since changes in levels of residential investment are more related to swings in the house price rather than the level of housing construction. It is a useful feature of the model if one of its goals is to predict house price movement. In this respect, the retained housing stock is a counterpart for land that is used in similar models with housing sector.\(^{14}\) Finally, there is a technology shock \(A_{x,t}\) specific to housing construction. The resulting output is the following:

\[
Y_{x,t} = (A_{x,t}n_{x,t})^{1-\alpha_{zk}-\alpha_{xx}-\alpha_{xh}}K_{x,t-1}^{\alpha_{zk}}X_t^{\alpha_{xx}}\bar{H}_t^{\alpha_{xh}}. \tag{1.19}
\]

The retained housing stock is defined simply as

\[
\bar{H}_t = \left(\Psi H_{t-1} + (1 - \Psi) \hat{H}_{t-1}\right) (1 - \delta_h) \Omega_t. \tag{1.20}
\]

Construction sector uses the retained housing stock for free—this assumption explains the existence of positive profit from construction sector \(\Pi_t\) in equilibrium, which belongs to the savers. This profit is very small, however.\(^{15}\) The profit of the sector is the following:

\[
\Pi_t = (A_{x,t}n_{x,t})^{1-\alpha_{zk}-\alpha_{xx}-\alpha_{xh}}K_{x,t-1}^{\alpha_{zk}}X_t^{\alpha_{xx}}\bar{H}_t^{\alpha_{xh}}P_t - W_{x,t}n_{x,t} - r_{x,t}K_{x,t-1} - X_t. \tag{1.21}
\]

Profit-maximization yields the prices for housing, labor, and capital stock:

\[
P_t = \frac{X_t}{\alpha_{xx}} \frac{1}{Y_{x,t}}, \quad \tag{1.22}
\]

\[
W_{x,t} = \frac{X_t}{\alpha_{xx}} \frac{1 - \alpha_{zk} - \alpha_{xx} - \alpha_{xh}}{n_{x,t}}, \tag{1.23}
\]

\[
r_{x,t} = \frac{X_t}{\alpha_{xx}} \frac{\alpha_{zk}}{K_{x,t-1}}. \tag{1.24}
\]

\(^{14}\)Neri and Iacoviello (2008) provide analogous justification for the inclusion of land in housing output function.

\(^{15}\)Rather than having zero housing-sector profit, it is important that housing stock rent does not affect the decision to purchase housing by the households. Also, the contractors do not pay housing rent when renovating the houses.
Notice that house price is negatively related to all factor inputs except for consumption good $X_t$. The presence of consumption good as one of the factors of housing construction allows for easy control of housing supply elasticity. Marginal products of labor and capital also positively depend on consumption good.

The housing sector resource constraint equates household purchases of housing stock net of aggregate housing stock retained from the previous period with the newly-constructed housing:

$$\Psi \dot{H}_t + (1 - \Psi) \ddot{H}_t - \ddot{H}_t = Y_{x,t}. \quad (1.25)$$

### 1.2.4 Shocks

Technologies $A_{y,t}$, $A_{x,t}$, and $A_{k,t}$ are non-stationary stochastic processes defined as

$$\text{for } i \in \{y, x, k\}, \quad \ln A_{i,t} = \ln A_{i,t-1} + \gamma_{i,t} + u_{i,t}, \quad u_{i,t} \sim N(0, \sigma_{u,i}^2), \text{ i.i.d.} \quad (1.26)$$

$$\gamma_{i,t} = (1 - \rho_i)\gamma_{i,t-1} + \rho_i \gamma_{i,t-1} + v_{i,t}, \quad v_{i,t} \sim N(0, \sigma_{v,i}^2), \text{ i.i.d.} \quad (1.27)$$

Technology $A_{i,t}$ includes a trend component $\gamma_{i,t}$ which is stationary around $\gamma_i$ and subject to shock $v_{i,t}$ labeled as persistent shock. The second component of technology growth is a transitory shock $u_{i,t}$. Since the technologies are non-stationary, I have to de-trend the model in order to solve it.\(^{16}\)

I consider three non-technological exogenous processes:

$$\ln \psi_t = \rho_\psi \ln \psi_{t-1} + (1 - \rho_\psi) \ln \psi + \epsilon_{\psi,t}, \quad \epsilon_{\psi,t} \sim N(0, \sigma_\psi^2), \text{ i.i.d.} \quad (1.28)$$

$$\ln \Omega_t = \rho_\Omega \ln \Omega_{t-1} + \epsilon_{\Omega,t}, \quad \epsilon_{\Omega,t} \sim N(0, \sigma_\Omega^2), \text{ i.i.d.} \quad (1.29)$$

$$\ln \nu_t = \rho_\nu \ln \nu_{t-1} + \epsilon_{\nu,t}, \quad \epsilon_{\nu,t} \sim N(0, \sigma_\nu^2), \text{ i.i.d.} \quad (1.30)$$

First, $\psi_t$ is the share of housing stock in household utility. It is a housing demand shock that shifts household preferences towards housing and affects the house price and the level

\(^{16}\)Details of de-trending and the de-trended system are provided in part A of the Appendix.
of residential investment. Second, Ωt is a shock to aggregate retained housing stock. It could be viewed as a shock to housing stock depreciation which changes the availability of existing housing. Finally, νt is a shock variable that makes households value current period’s consumption and housing services differently and affects their inter-temporal choice.

The equilibrium is defined dynamically by equations (1.7)–(1.30) and a set of market-clearing conditions: (1 − Ψ) ˆS_t = Ψ B_t; W_{y,t} = W_{x,t}; n_{y,t} + n_{x,t} = 1; (1 − Ψ) ˆK_{y,t} = K_{y,t}; (1 − Ψ) ˆK_{x,t} = K_{x,t}; (1 − Ψ) ˆΠ_t = Π_t.

1.2.5 Model Solution

Before introducing imperfect knowledge, it is convenient to outline the solution of the model. Because of the technology processes, the model is non-stationary. It can be approximated by the following linear state-space form:

\[ z_t = \Phi_0 + \Phi_1 s_{1,t} + \Phi_2 s_{2,t} + \Phi_3 t; \]
\[ s_{1,t} = A_1 s_{1,t-1} + B_1 \varepsilon_t; \]
\[ s_{2,t} = s_{2,t-1} + B_2 \varepsilon_t. \]

In this formulation, \( z_t \) is the vector of logged observable variables. Vector \( s_{1,t} \) represents the set of log-deviations from the balanced growth path. In other words, the system (1.32) is the de-trended log-linearized version of the model. Vector \( \varepsilon_t \) contains all the shocks to exogenous processes, including the transitory and persistent shocks to technologies: \( \varepsilon_t = (\{\varepsilon_{j,t}\}, \{u_{i,t}\}, \{v_{i,t}\})' \). Notice that what matters for the de-trended system are not the levels of technology \( A_{y,t} \), \( A_{k,t} \), and \( A_{x,t} \), but their growth rates. Given \( i \in \{y, k, x\} \), I redefine equations (1.26) and (1.27) in terms of log-deviations from the steady-state growth rates in order to include them in the system (1.32):

\[ \tilde{g}_{i,t} = \tilde{\gamma}_{i,t} + u_{i,t}; \]
\[ \tilde{\gamma}_{i,t} = \rho_t \tilde{\gamma}_{i,t-1} + v_{i,t}, \]
where \( g_{i,t} = A_{i,t}/A_{i,t-1} \) is the growth rate, so that \( \tilde{g}_{i,t} = \ln A_{i,t} - \ln A_{i,t-1} - \gamma_i \) is the log-deviation of the growth rate from its steady state; and \( \tilde{\gamma}_{i,t} = \gamma_{i,t} - \gamma_i \) is the deviation of the growth rate’s persistent component. Consequently, the state vector \( s_{1,t} \) contains these variables: \( s_{1,t} = (\ldots, \{\tilde{g}_{i,t}\}, \{\tilde{\gamma}_{i,t}\})' \). Note that \( s_{1,t} \) is stationary, which justifies the use of log-linearization for this component of the model. Vector \( \Phi_0 \) captures the steady-state values of the de-trended observable variables. The last two terms capture the non-stationary stochastic and deterministic components of technology processes. In the Appendix, section A.1.2 explains the de-trending procedure; section A.1.3 provides the complete de-trended system; section A.1.4 provides its log-linearized version correspondent to equation (1.32); and section A.1.6 explains how different components of the system (1.31) capture the technology processes.

### 1.2.6 Imperfect Knowledge and Learning

The model with perfect knowledge implies that agents can perfectly observe both persistent and transitory innovations to exogenous processes. In case of imperfect knowledge, agents only observe the total level of productivity—or, equivalently, its growth rate—but the shocks \( u_{i,t} \) and \( v_{i,t} \) and the persistent component \( \gamma_{i,t} \) are unobservable. For an example, the agents can observe a large increase in technology \( A_{i,t} \) but cannot immediately tell if the higher growth is due to the persistent component \( v_{i,t} \) and, thus, if the growth rate will be higher in the future as well; or if it is a one-period increase due to the transitory component \( u_{i,t} \). To resolve this uncertainty, the agents have to wait and observe the growth rate for several periods in order to gradually learn about the nature of the shock. I assume that the agents use a simple linear steady-state Kalman filter for this purpose. The motivation behind the steady-state filter is that the agents have observed a long history of productivity in the economy and have an idea about the transitory and persistent shocks, their volatility and persistence. For a technology \( A_{i,t} \), the agents are assumed to know the values \( \sigma_{v_i}, \sigma_{ui}, \rho_i, \) and \( \gamma_i \). An interesting question (which I address in section 1.4.3) is whether the values that the agents know correspond to the actual processes, or if they are different from the
truth—but the point is that the agents are assumed to rely on their belief and implement the steady-state filter.

Define \( \hat{\gamma}_{i,t} = E(\tilde{\gamma}_{i,t} | \tilde{g}_{i,0}, \ldots, \tilde{g}_{i,t}) \) as the inferred value of the technological growth’s persistent component at time \( t \) given all available observations of the growth rate up to period \( t \). Then, given the equations (1.34) and (1.35), the standard result is the following Kalman-updating equation:

\[
\hat{\gamma}_{i,t} = \lambda_i \tilde{g}_{i,t} + (1 - \lambda_i) \rho_i \hat{\gamma}_{i,t-1}.
\]

This expression summarizes the way households learn about the value of persistent component of technological growth by means of the steady-state Kalman filter. Parameter \( \lambda_i \) is the steady-state Kalman gain:

\[
\lambda_i = \frac{d_i - (1 - \rho_i^2) + \sqrt{(1 - \rho_i^2)^2 + d_i^2 + 2(1 + \rho_i^2)d_i}}{2 + d_i - (1 - \rho_i^2) + \sqrt{(1 - \rho_i^2)^2 + d_i^2 + 2(1 + \rho_i^2)d_i}}, \quad (1.36)
\]

where \( d_i = \sigma_{v,i}^2 / \sigma_{u,i}^2 \) relates the volatilities of persistent and transitory shocks. The Kalman gain positively depends on \( d_i \) and \( \rho_i \). Intuitively, if the persistent shock is more volatile compared to the transitory shock or if the persistent component has a higher autocorrelation coefficient, the agents infer that a sequence of higher growth rates is more likely to be the result of a shock to persistent component rather than a sequence of transitory shocks.

Figure 1.2.6 demonstrates the process of learning. There is a shock to technology that increases the growth rate \( g_{i,t} \). The graph on the left shows the case of persistent shock. Initially, the agents do not completely realize that the shock is persistent (there is a chance that it is a transitory shock), so the inferred value of the persistent component is lower than the actual value. As agents keep observing the evolution of the growth rate, they gradually learn that the shock has been persistent, since the observed path of the growth rate is far more likely to be the result of one persistent shock rather than a series of transitory

---

17The Kalman-updating equations are derived in section A.1.5 of the Appendix.
shocks. The two lines eventually converge. A relatively higher variance of transitory shock (low $d_i$) corresponds to slower learning, since a sequence of positive transitory shocks that can explain the observed growth rate path becomes more likely. On the right-side graph, the shock is to the transitory component, and the learning mechanics are analogous. The persistent component does not change, but initially, agents realize that the shock might be persistent. After observing zero growth rates in all subsequent periods, the agents eventually learn that the shock is transitory.

In order to impose the case of imperfect knowledge on the model, I take the linear system (1.32) and replace the actual values of persistent technology components and shocks to technologies with the ones inferred through Kalman filtering, so that the state vector becomes $\hat{s}_{1,t} = (\ldots, \{\hat{g}_{i,t}\}^i, \{\hat{\gamma}_{i,t}\}^i)'$ and the vector of shocks becomes $\hat{\varepsilon}_t = (\{\epsilon_{j,t}\}^j, \{\hat{u}_{i,t}\}^i, \{\hat{v}_{i,t}\}^i)'$.

Furthermore, for each of the technology processes, I add a set of equations that relate the Kalman-filtered estimates and the actual values of components of technology processes:

$$\hat{v}_{i,t} = \lambda_i (\hat{\gamma}_{i,t} + u_{i,t}) - \lambda_i \rho_i \hat{\gamma}_{i,t-1};$$
\[
\hat{u}_t = (1 - \lambda_i)(\hat{\gamma}_{i,t} + u_{i,t}) - (1 - \lambda_i)\rho_i \hat{\gamma}_{i,t-1};
\]
\[
\hat{\gamma}_{i,t} = \rho_i \hat{\gamma}_{i,t-1} + v_t.
\]

### 1.3 Estimation

The empirical exercise outlined below seeks to establish how technological shocks contribute to the observable data pertinent to aggregate economy and to the market for housing in particular, and how the assumption of imperfect knowledge affects this contribution. For this purpose, I perform the classical maximum likelihood estimation of the model under both perfect and imperfect knowledge. The parameters estimated by means of MLE are all the parameters describing the exogenous processes (shock variances, autoregressive coefficients, steady-state technological growth rates); all the other parameters are calibrated so that the steady state of the model matches certain empirical targets. Given the observed data \( Z_T = \{z_t\}_{t=1}^T \) and a parametrization \( \theta \), I can compute the log-likelihood function \( l(Z_T|\theta) = \ln \Pi_{t=1}^T \Pr(z_t|Z_{t-1}, \theta) \) by means of sequential Kalman filtering. In order to find the vector of parameters \( \theta^*_\mathcal{M} \) that maximizes the log-likelihood function for the models with perfect and imperfect knowledge \( \mathcal{M} \in \{P, I\} \), I use a combination of Newton methods.\(^\text{18}\)

#### 1.3.1 Relating the Model and Data

The aggregate resource constraint is a combination of the borrower’s and saver’s budget constraints (1.12) and (1.7):

\[
\begin{align*}
\underbrace{\Psi C_t + (1 - \Psi) \hat{C}_t}_{\text{Household consumption}} + \underbrace{\frac{K_{y,t}}{A_{y,t}} - (1 - \delta_y)\frac{K_{y,t-1}}{A_{y,t-1}} + K_{x,t} - (1 - \delta_x)K_{x,t-1} + }_{\text{Non-residential investment}}
\end{align*}
\]

\(^{18}\)I start the search for the parameter vector \( \theta^*_\mathcal{M} \) using my own version of the BFGS routine; then, I use Chris Sims’ optimization algorithm (http://sims.princeton.edu/yftp/optimze/) to ’polish’ the result.
\[
\Psi H_{t-1} P_t (1 - \delta_h) \mu E[\omega | \omega < \bar{\omega}_t] + P_t (\Psi H_t + (1 - \Psi) \hat{H}_t - \bar{H}_t) = Y_t + Y_{x,t} P_t \tag{1.37}
\]

This equation conveniently relates the model to the four key variables taken from the data: aggregate consumption \(AC_t\), which is the sum of household consumption and the cost of default; non-residential investment \(IK_t\); and residential investment \(IH_t\). The fourth observable variable is the house price, \(P_t\). The data set includes the four quarterly series \(Z_T = \{AC_t, IK_t, IH_t, P_t\}_{t=1}^T\) and spans 1975–2013. The full description of the data is provided in section A.2 of the Appendix.

1.3.2 Calibration

The vector of parameters \(\theta\) that is estimated by MLE describes the exogenous processes: \(\theta = \{\rho_i, \gamma_i, \sigma_i\}\). I calibrate the rest of the parameters to match the steady state of the model with a corresponding number of empirical targets.\(^{19}\) Table 1.1 summarizes the calibration.

Saver’s discount factor is set to match the quarterly interest rate, which corresponds to the real expected return on capital in my model. For 1975–2006, I have estimated the average quarterly 3-Month Treasury Bill rate adjusted for expected inflation to be 0.5%.\(^{20}\) Data show that the equity risk-premium over the T-Bills was 1.7% in quarterly terms for the same period;\(^{21}\) and, for 30-year corporate bonds rated Aaa by Moody’s, the risk-premium over the Treasury bonds with the same maturity was 0.2%.\(^{22}\) These values imply a reasonable range for the real interest rate between 0.7% and 2.2% per quarter. I set \(\hat{\beta} = 0.9888\) to match the target of 1.5%. Given \(\hat{\beta}\), the borrower’s discount factor \(\beta\), cost of default parameter \(\mu\), and variance of idiosyncratic housing stock shock \(\sigma_\omega\) are

---

\(^{19}\)The technology growth rates \(\gamma_i, i \in \{y, k, x\}\) obtained via MLE affect the steady state of the model, which slightly complicates the calibration. However, they are quite insensitive to calibration, so I use the growth rates obtained after a few initial MLE runs to calibrate the other parameters.

\(^{20}\)Inflation is the growth in CPI for All Urban Consumers: All items less shelter (BLS). Expected inflation is the OLS prediction of the quarterly inflation rate based on four lagged values.

\(^{21}\)Ibbotson Risk Premia Over Time Report, 2013

\(^{22}\)Release H.15 by the Federal Reserve
Table 1.1: Calibrated parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\beta} )</td>
<td>0.9888</td>
<td>Saver’s discount factor</td>
</tr>
<tr>
<td>( \psi )</td>
<td>0.17</td>
<td>Share of housing in utility function</td>
</tr>
<tr>
<td>( \alpha_y )</td>
<td>0.25</td>
<td>Share of capital in consumption good production</td>
</tr>
<tr>
<td>( \alpha_{xk} )</td>
<td>0.1</td>
<td>Share of capital in housing construction</td>
</tr>
<tr>
<td>( \alpha_{xh} )</td>
<td>0.1</td>
<td>Share of housing stock in housing construction</td>
</tr>
<tr>
<td>( \alpha_{xx} )</td>
<td>0.1</td>
<td>Share of consumption good in housing construction</td>
</tr>
<tr>
<td>( \delta_y )</td>
<td>0.02</td>
<td>Depreciation rate of consumption capital</td>
</tr>
<tr>
<td>( \delta_x )</td>
<td>0.025</td>
<td>Depreciation rate of housing construction capital</td>
</tr>
<tr>
<td>( \delta_h )</td>
<td>0.015</td>
<td>Depreciation rate of housing stock</td>
</tr>
<tr>
<td>( N )</td>
<td>1.0</td>
<td>Population size</td>
</tr>
<tr>
<td>( \Psi )</td>
<td>0.6418</td>
<td>Borrowers’ share in population</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.9587</td>
<td>Borrower’s discount factor</td>
</tr>
<tr>
<td>( \sigma_{\omega} )</td>
<td>0.1902</td>
<td>Standard error of idiosyncratic shock to house size</td>
</tr>
<tr>
<td>( \mu )</td>
<td>0.1172</td>
<td>Cost of mortgage foreclosure</td>
</tr>
</tbody>
</table>

chosen jointly to match the loan-to-value ratio, default rate, and mortgage premium. The annual mortgage premium is chosen to be 1.5% based on the average 30-year mortgage premium over the 30-year Treasury bonds for 1977–2006.\(^{23}\) Following the literature, the target default rate is chosen to be 2%, which is the average delinquency rate for residential estate loans for the decade preceding 2007, according to the Federal Reserve. Neri and Iacoviello (2008) report the average loan-to-value ratio to be 76% between the years 1973 and 2006.\(^{24}\) At the rate of default of 2%, such high loan-to-value ratio would correspond to an extremely low borrower’s discount factor. I set a more moderate target of 65%. The resulting values of \( \mu \) and \( \sigma \) are in line with the literature;\(^{25}\) the borrower’s discount factor is quite low, which is to say that the model implies a very impatient borrower in order for her to accept the mortgage that has a high chance of costly default.

Capital depreciation in consumption good sector is set at \( \delta_y = 0.02 \). Together with

\(^{23}\)Ibid.
\(^{24}\)Finance Board’s Monthly Survey of Rates and Terms on Conventional Single-Family Non-farm Mortgage Loans (table 19)
\(^{25}\)Forlati and Lambertini (2011) set \( \sigma = 0.2 \) and \( \mu = 0.12 \). The cost of debt parameter \( \mu \) is quite conservative; for a detailed discussion of foreclosure discounts, see Campbell et al. (2009).
capital share in consumption good production function, it helps to set the share of non-residential investment in GDP. Housing sector capital stock is relatively too small for it to have a significant impact on the aggregate investment share. Housing sector capital depreciation is set at $\delta_h = 0.025$ to reflect the fact that capital stock has shorter life in construction sector. The share of capital is set at $\alpha_y = 0.25$ in consumption good sector and $\alpha_{xk} = 0.1$ in construction sector, which is less capital-intensive. These shares are considerably lower than the conventional one third due to the definition of GDP which only includes private consumption and private fixed investment. Davis and Heathcote (2005) use the NIPA Input-Output tables to estimate the share of capital to be 0.13 in construction, 0.24 in services, and 0.31 in manufacturing. The shares of consumption good and housing stock in construction output are chosen to be $\alpha_{xx} = 0.1$, $\alpha_{xh} = 0.1$. These parameter values are according to Davis and Heathcote (2005), which set these shares for consumption good and land. I am using the empirically estimated share of land to measure the contribution of housing stock because housing stock used in construction acts as a counterpart to land in my model.\(^{26}\)

The share of housing in the utility is set in combination with housing depreciation rate to match the share of residential investment in GDP and the ratio of housing stock value to GDP: $\psi = 0.17$, $\delta_h = 0.01$. The share of borrowers in total population equals the share of homeowners that had mortgages in 2006:\(^{27}\) $\Psi = 0.6418$.

Table 1.2 shows that the steady-state of the model is close to the empirical targets that I have set. Given the calibration, I estimate the remaining parameters $\theta = \{\gamma_i, \rho_i, \sigma_i\}$ via the maximum-likelihood estimation.

\(^{26}\)See section 1.2.3 for details
\(^{27}\)Consumer Expenditure Survey, 2006-2010, Bureau of Labor Statistics
Table 1.2: Steady-state performance of the model

<table>
<thead>
<tr>
<th>Measure</th>
<th>Model</th>
<th>Target</th>
<th>Target source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarterly interest rate, %</td>
<td>1.51</td>
<td>1.50</td>
<td>Data, Federal Reserve</td>
</tr>
<tr>
<td>$IK/GDP$, %</td>
<td>13.60</td>
<td>15.00</td>
<td>Data, BEA</td>
</tr>
<tr>
<td>$IH/GDP$, %</td>
<td>6.14</td>
<td>6.00</td>
<td>Data, BEA</td>
</tr>
<tr>
<td>$p \times H/GDP$</td>
<td>1.14</td>
<td>1.36</td>
<td>Neri and Iacoviello (2008)</td>
</tr>
<tr>
<td>$K/GDP$, non-residential</td>
<td>2.31</td>
<td>2.05</td>
<td>Neri and Iacoviello (2008)</td>
</tr>
<tr>
<td>$K/GDP$, residential</td>
<td>0.04</td>
<td>0.04</td>
<td>Neri and Iacoviello (2008)</td>
</tr>
<tr>
<td>Annual mortgage premium, %</td>
<td>1.50</td>
<td>1.50</td>
<td>Data, Federal Reserve</td>
</tr>
<tr>
<td>Mortgage default rate, %</td>
<td>2.00</td>
<td>2.00</td>
<td>Data, Federal Reserve</td>
</tr>
<tr>
<td>Loan-to-value ratio, %</td>
<td>64.60</td>
<td>65.00</td>
<td>Data, FHFA</td>
</tr>
</tbody>
</table>

1.4 Results

1.4.1 Maximum-Likelihood Estimation

1.4.1.1 Parameter estimates

Table 1.3 presents the parameter estimates with associated standard errors\(^\text{28}\) for the cases of perfect and imperfect knowledge. While the shock variances do not describe their relative importance, the table reveals a few notable facts. First, the processes for which ambiguity about the persistence of technology shocks matters are capital and consumption technologies. Under imperfect knowledge, these are the only processes with both persistent and transitory shocks significant. At the same time, agents always know that the transitory component is driving the construction technology because the persistent component is nonexistent. Second, capital technology shocks have significant variances only under imperfect knowledge. Third, imperfect knowledge makes both transitory and persistent components of consumption technology significant, while perfect knowledge renders the transitory shock insignificant. I look into the reasons behind these facts below.

In both cases, housing demand shock $\psi_i$ is insignificant. Retained housing supply shock

\(^{28}\text{Standard errors are evaluated using delta-method and the Hessian that is numerically estimated for the likelihood function.}\)
**Table 1.3: Estimated parameters**

### Case I: Imperfect Knowledge

<table>
<thead>
<tr>
<th></th>
<th>$\gamma_i$</th>
<th>$\rho_i$</th>
<th>$\sigma_{ui}$</th>
<th>$\sigma_{ei}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Technological shocks</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Consumption production ($g_y$)</td>
<td>0.0069</td>
<td>0.4945</td>
<td>0.0054</td>
<td>0.0054</td>
</tr>
<tr>
<td>(0.0009)</td>
<td>(0.0725)</td>
<td>(0.0009)</td>
<td>(0.0009)</td>
<td>(0.0009)</td>
</tr>
<tr>
<td>Capital creation ($g_k$)</td>
<td>-0.0095</td>
<td>0.8916</td>
<td>0.0123</td>
<td>0.0022</td>
</tr>
<tr>
<td>(0.0015)</td>
<td>(0.0401)</td>
<td>(0.0031)</td>
<td>(0.0003)</td>
<td></td>
</tr>
<tr>
<td>Construction ($g_x$)</td>
<td>-0.0032</td>
<td>0.9900</td>
<td>0.0199</td>
<td>0.0000</td>
</tr>
<tr>
<td>(0.0016)</td>
<td>(x.xxxx)$^\dagger$</td>
<td>(0.0012)</td>
<td>(0.0000)</td>
<td></td>
</tr>
<tr>
<td><strong>Non-technological shocks</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inter-temporal preference ($\nu$)</td>
<td>0.9900</td>
<td>0.0000</td>
<td>(x.xxxx)$^\dagger$</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>Housing demand ($\psi$)</td>
<td>0.9900</td>
<td>0.0000</td>
<td>(x.xxxx)$^\dagger$</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>Retained housing supply ($\Omega$)</td>
<td>0.0000</td>
<td>0.0184</td>
<td>(0.0000)</td>
<td>(0.0011)</td>
</tr>
</tbody>
</table>

### Case II: Perfect Knowledge

<table>
<thead>
<tr>
<th></th>
<th>$\gamma_i$</th>
<th>$\rho_i$</th>
<th>$\sigma_{ui}$</th>
<th>$\sigma_{ei}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Technological shocks</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Consumption production ($g_y$)</td>
<td>0.0060</td>
<td>0.3791</td>
<td>0.0000</td>
<td>0.0074</td>
</tr>
<tr>
<td>(0.0009)</td>
<td>(0.0328)</td>
<td>(0.0000)</td>
<td>(0.0005)</td>
<td></td>
</tr>
<tr>
<td>Capital creation ($g_k$)</td>
<td>-0.0079</td>
<td>0.3579</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>(0.0014)</td>
<td>(0.4741)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td></td>
</tr>
<tr>
<td>Construction ($g_x$)</td>
<td>-0.0099</td>
<td>0.9900</td>
<td>0.0204</td>
<td>0.0007</td>
</tr>
<tr>
<td>(0.0016)</td>
<td>(x.xxxx)$^\dagger$</td>
<td>(0.0012)</td>
<td>(0.0001)</td>
<td></td>
</tr>
<tr>
<td><strong>Non-technological shocks</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inter-temporal preference ($\nu$)</td>
<td>0.9900</td>
<td>0.0236</td>
<td>(x.xxxx)$^\dagger$</td>
<td>(0.0026)</td>
</tr>
<tr>
<td>Housing demand ($\psi$)</td>
<td>0.9900</td>
<td>0.0000</td>
<td>(x.xxxx)$^\dagger$</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>Retained housing supply ($\Omega$)</td>
<td>0.2832</td>
<td>0.0105</td>
<td>(0.0505)</td>
<td>(0.0014)</td>
</tr>
</tbody>
</table>

$^\dagger$ The values of autoregressive coefficient were restricted to not exceed 0.99—this restriction is only effective for the housing construction technology and inter-temporal preference in case of perfect knowledge.
$\Omega_t$ is either a white noise or a very nonpersistent process. Inter-temporal preference shock $\nu_t$ is only significant under perfect knowledge. It seems that the contribution of capital technology shock is replaced by that of the inter-temporal preference shock under perfect knowledge; both shocks affect the saver’s decision to invest. Inter-temporal preference shock is estimated to be extremely persistent; I put a cap of $\rho_\nu = 0.99$ on its autoregressive coefficient. Wen (2006) explains that a positive shock to inter-temporal preference variable, when it is transitory, makes households value current utility relatively much more, so they prefer to consume more and reduce investment; the result is that consumption and investment are negatively related. A more persistent shock also increases the value of future utility by more and allows for better co-movement between consumption and investment.

### 1.4.1.2 A Note on Comparative performance

The model with imperfect knowledge ($I$) performs better in terms of likelihood: the achieved log-likelihood is $l(Z_T|\theta^*_I, I) = 1637.9$; for the case of perfect knowledge ($P$), the value is $l(Z_T|\theta^*_P, P) = 1617.9$, where $\theta^*$ is the vector of parameters that maximizes the likelihood function. While this is a considerable difference for log-likelihood values, the two numbers are not directly comparable, since the two models have different structures and do not simply correspond to two alternative parameter specifications. A more appropriate way to compare the two models would be to specify a prior over the parameter space $p(\theta|\mathcal{M})$ for the two models of perfect and imperfect knowledge $\mathcal{M} \in \{P, I\}$ and use numerical methods to compute the marginal likelihood $p(Z_T|\mathcal{M}) = \int p(Z_T|\theta)p(\theta|\mathcal{M})d\theta$. Then, it is straightforward to compare the two models in terms of posterior odds in favor of the model with imperfect knowledge:

$$PO = \frac{p(Z_T|I)}{p(Z_T|P)}.$$

I conduct this experiment and outline the results in Chapter 2 of the dissertation.
1.4.2 Impulse Responses

The goal of this subsection is to shed light on how imperfect knowledge affects the dynamics of the model. I limit the discussion to responses to shocks that are significant under imperfect knowledge.

1.4.2.1 Consumption technology shocks

Panel A of figure 1.3 shows impulse responses of key variables due to a negative one-standard-deviation shock to persistent component of consumption technology growth $g_{y,t}$. Consumption good is used to pay for capital and new housing stock, so a less productive consumption good sector results in lower consumption and investment, which all eventually stabilize at 1.07 percent below the pre-shock balanced growth path. Construction capital stock initially increases because construction sector becomes relatively more productive, but eventually, scarcer consumption good makes construction capital stock fall. Because consumption sector becomes relatively less productive than construction, housing demand declines together with house prices, leading to a higher mortgage default rate and lower mortgage lending. Notice how imperfect knowledge delays the responses: initially, people believe in quick recovery, so consumption, housing demand, and house prices remain higher; these variables adjust as households gradually recognize the persistent shock.

Overall, imperfect knowledge delays the responses of house price, consumption, and mortgage default rate. This is the kind of protraction which Burnside et al. (2011) argue to be hard to achieve when the household beliefs are homogeneous. In part, they prove to be right: this protraction is very limited under the estimated parametrization, and the responses for the two models converge after 2–3 quarters. Kalman gains are defined by parameters of technology processes according to equation (1.36), and consumption technology process is such that agents learn about the nature of the shock very quickly. Still, the persistent shock has an ability to explain a lot of longer-term dynamics in observable variables, since its contribution builds up over time while the immediate impact is subdued.
Figure 1.3: Impulse-responses. Panel A: persistent consumption technology shock, $v_y$. Panel B: transitory consumption technology shock, $u_y$. Percentage deviations of non-detrended variables from the balanced growth path due to one-standard-deviation negative shock. Solid lines represent the case of certainty; dashed lines represent uncertainty. For default rate, the values are absolute, compared with their steady-state values (dotted lines). Numbers near the arrows show where the deviations will stabilize after 400 quarters.

Notably, imperfect knowledge creates additional redistribution of wealth between borrowers and savers during a recession. Immediately after the persistent shock, borrowers are too optimistic about the future house price and bet on it by obtaining large mortgages and making large housing purchases. These decisions turn out to be bad ex post: in the periods following the shock, unexpectedly low house price results in higher default rate and reduces the borrowers’ net worth, consumption, and housing demand. Meanwhile, savers are able to purchase cheaper housing stock and use the extra savings to maintain higher consumption. Section A.3 of the Appendix contains details on wealth redistribution.
between the household groups.

Panel B of figure 1.3 depicts the case of transitory shock. Essentially, the logic behind the responses is similar. Imperfect knowledge amplifies the immediate responses, since agents fear a long recession; which is the opposite to the case of a persistent shock. Compared to the case of perfect knowledge, such variability in responses makes both persistent and transitory shocks versatile tools to describe the dynamics of the observed variables that complement each other: persistent shock explains a lot of low-frequency dynamics, and transitory shock is better at explaining short-term fluctuations.

1.4.2.2 Capital technology shocks

Reaction of the model to capital technology shocks illustrates yet another dimension along which imperfect knowledge can improve the dynamics: co-movement of key observable variables. Panels A and B of figure 1.4 show responses due to a negative shock to capital technology’s persistent and transitory component, respectively. Consider the transitory shock first. It makes consumption sector’s capital more expensive in current period. In case of perfect knowledge, savers initially reduce consumption-capital investment and choose to allocate their wealth to consumption, construction capital, and housing purchases. Because of higher housing demand from savers, house price increases, default rate falls, and more expensive housing requires that borrowers get larger mortgages. Eventually, because of lower stock of consumption sector capital, output in consumption sector declines, together with aggregate consumption, house price, and capital investment in both sectors. Of course, consumption-sector capital is affected by this shock the most in the long run. Note that initially, the response is such that aggregate consumption, house price, and residential investment increase while capital investment declines.

The picture is the opposite for the persistent shock. The cost of consumption capital does not only increase in current period but keeps growing at a higher rate in subsequent periods. In case of perfect knowledge, it means that investors not only pay more for consumption-sector capital; they expect it to be worth even more next period. In ef-
Panel A

Figure 1.4: Impulse-responses. Panel A: persistent capital technology shock, $v_k$. Panel B: transitory capital technology shock, $u_k$. Percentage deviations of non-detrended variables from the balanced growth path due to one-standard-deviation negative shock. Solid lines represent the case of certainty; dashed lines represent uncertainty. For default rate, the values are absolute, compared with their steady-state values (dotted lines). Numbers near the arrows show where the deviations will stabilize after 400 quarters.

Perfect, the return on one unit of consumption good spent on consumption-sector capital rises, and savers increase investment into consumption sector at the cost of consumption, housing purchases, and construction-sector capital. The initial response is that aggregate consumption, house price, and residential investment decline while non-residential investment rises. Eventually, all four observable variables stabilize below the pre-shock balanced growth path.

Imperfect knowledge improves the co-movement of capital investment with other variables due to capital technology shocks. This is especially evident for persistent shock,
where a lot of counter-factual dynamics of the perfect-knowledge model in the first few initial periods upon shock is washed out. Capital technology shocks can be an important driver of long-term dynamics of capital investment, and imperfect knowledge helps eliminate the counter-cyclicality of this variable. Huang et al. (2009) and Edge et al. (2007) have similar findings with respect to learning and co-movement. Under perfect knowledge, the contribution of capital technology shocks is replaced with that of inter-temporal preference shock $\nu_t$, which does not generate permanent deviations from the balanced growth path and which, speaking in terms of the likelihood, is a costlier explanation of low-frequency movement in capital investment.

### 1.4.2.3 Construction technology and housing supply shocks

Panel B of figure 1.5 demonstrates the impact of the shock to aggregate housing stock retained from the previous period. It could be viewed as a shock to depreciation or an (negative) overbuilding shock. Lower retained housing stock implies higher house price. Higher house price does not fully compensate for the effect of the shock on the value of the housing stock, so the default rate increases. Expensive housing, as well as higher return on construction capital, shifts investors’ preference towards construction capital; capital investment increases, as well as residential investment, while consumption falls. This shock is specific to the market for housing and may explain a lot of dynamics in residential investment.

The last shock that is significant under imperfect knowledge is the transitory construction technology shock (panel A of figure 1.5). Lower productivity in construction sector reduces the supply of housing and increases the house price and mortgage borrowing; the mortgage default rate falls. Construction capital is less productive; it drives the total capital investment down. Despite higher house prices, the decline in construction sector’s output causes residential investment to decline as well; it helps that households substitute

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29Chatterjee and Eyigungor (2011) model an overbuilding shock with similar implications for house prices and default rate.
Figure 1.5: Impulse-responses. Panel A: transitory construction technology shock, $u_x$. Panel B: housing supply shock, $u_\Omega$. Percentage deviations of non-detrended variables from the balanced growth path due to one-standard-deviation negative shock. Solid lines represent the case of certainty; dashed lines represent uncertainty. For default rate, the values are absolute, compared with their steady-state values (dotted lines). Numbers near the arrows show where the deviations will stabilize after 400 quarters.

away from expensive housing towards consumption. Mortgages become a relatively more lucrative saving option due to lower default rates and lower return on construction capital. Note that house price is the only variable that deviates from the balanced growth path permanently. This shock has a large and lasting effect on house price and can explain a lot of its movement.

Unfortunately, the model with imperfect knowledge renders the persistent component of construction technology insignificant, while it could be the source of more persistent dynamics of house price. The implication (confirmed below) is that the model with imper-
fect knowledge does not generate much additional inertia in house price compared to the model with perfect knowledge. To resolve this issue, I consider the case when the Kalman gain specific to construction technology $\lambda_x$ is not tied to the parameters that describe this technology process; rather, it is an extra parameter that is estimated using MLE.

1.4.3 Unrestricted Kalman gains

The assumption of imperfect knowledge can improve the model’s performance even further if I do not assume that the Kalman gain used to infer about the transitory and persistent components of technological growth is tied to the properties of the technology process (via equation (1.36)). Instead, I assume that it is a free parameter that needs to be estimated. Intuitively, it implies that agents are not only unable to observe whether the source of technological growth is transitory or persistent; they also do not know how the technology actually evolves. More precisely, they have an idea of what the technological process might be (i.e., they assume some values $\{\sigma_{ui}, \sigma_{vi}, \rho_i\}$), but it does not necessarily correspond to the actual process. It is likely that agents in the economy do not have a precise idea about the underlying technological processes that drive the economy. For example, Fuster et al. (2010) argue that agents, when forming forecasts, rely excessively on the most recent observations.

I only consider the unrestricted Kalman gain for construction-specific technology, for two reasons. First, the assumption that households do not know the true process that drives the house price is in line with the argument that housing market participants are less informed about the economy than capital investors. Construction-specific technology accounts for much of house price dynamics, unlike consumption- and capital-specific technologies that mostly affect consumption and capital investment. And second, I find it more illustrative to study the change of one Kalman gain in isolation. Of all three technologies, the Kalman gain for construction technology is the best choice, because it is the technology that contributes to house prices the most. After all, the goal of the whole exercise is to better explain the house price dynamics.
The assumption has strong implications. Table 1.4 summarizes the results of MLE. The log-likelihood for the model with imperfect knowledge and untied Kalman gain is 1657.5; which is higher than 1637.9 and 1617.9 for the models with restricted Kalman gains. When the Kalman gain is a free parameter, it is estimated to be zero ($\lambda_x = 0$): agents completely ignore the chance that the changes to the rate of construction-technology growth can be persistent. For example, in case of a negative persistent shock, the agents keep making over-optimistic forecasts until the rate of technological growth converges to its steady-state level. Notice how the variance of persistent component is relatively large under untied Kalman gain ($\sigma_{ux} = 0.0174$): the persistent component becomes a powerful explanatory variable in case of agents’ stubborn reluctance to update their forecasts.

Such slow ‘learning’ (or absence of such) in case of persistent shocks implies that house price dynamics are more protracted. To confirm this statement, I plot spectral densities of the observable variables’ first differences derived for the three models and compare them to less parametric estimates. In figure 1.6, the less parametric estimates include an estimate based on VAR\textsuperscript{30} and non-parametric kernel density estimates constructed for each individual series. First, notice that with Kalman gain tied to technology process, the model under imperfect knowledge is better at generating low-frequency dynamics for capital investment and, to a much smaller extent, for consumption—this is mostly because capital technology shocks are significant only under imperfect knowledge. As for house price and residential investment, imperfect knowledge alone does not contribute to their...
low-frequency dynamics. If anything, the spectral density of house price is slightly heavier on the low-frequency range under perfect knowledge, because under perfect knowledge, the persistent construction technology component is estimated to be significant. The picture dramatically changes after the Kalman gain is untied from the technology process. It is especially evident for residential investment and house price: because of volatile persistent component of construction technology, and because agents fail to learn about it, the variables specific to housing market gain a lot of density in low-frequency range and become

Figure 1.6: Spectral densities of the first-difference series normalized by the unconditional variance. The horizontal axes measure cycles per quarter. Spectral density is calculated for the models using the deep parameters obtained from MLE and compared against less parametric estimates: an estimate based on VAR, and completely non-parametric estimates for individual differenced series based on covariances.
much more resemblant of the data. There is also an evident gain in the low-frequency range of the spectral densities of consumption and capital investment.

Figure 1.7 exemplifies how gains in the low-frequency range of house price’s spectral density can transform into higher value of the likelihood function. As explained above,
technology shocks can account for most of long-run dynamics of the house price, which is shown on the bottom panel of the figure. As the top two panels show, it is construction-specific technology that can account for movements of house price that correspond to low-frequency domain. Notice that contribution of technologies to house-price dynamics is similar for all three models. While the price paths predicted by technological innovations are similar, the cost of these paths in terms of the likelihood function can be different. In terms of the likelihood, it is less costly to explain the protracted dynamics of house price by means of a small number of shocks that generate protracted responses rather than by a sequence of shocks that have transitory impact on house price.

1.5 Conclusion

There are two frequently discussed channels through which imperfect knowledge and learning allow general equilibrium models to better explain the observed data. First, they create a protracted evolution of economic variables; and, second, they may improve the co-movement between economic variables. I extend this mechanism to the model with endogenous market for housing and show that the same results apply to the housing market variables. It is particularly applicable to house prices, which experience long periods of steady growth and decline. The key point is that learning is a tool that helps account for the protracted dynamics of housing market variables, and can improve the predictive power of general equilibrium models with housing.

In line with Neri and Iacoviello (2008), I show that technology processes can account for most of long-run dynamics of observable variables, including house prices. I assert that persistent technological shocks are important for periods of consistent growth or decline in the variables. They become even more important when there is imperfect knowledge about the persistence of shocks and learning, which creates additional protraction and may improve co-movement.

To further the argument, I untie the learning process from the features of technology
processes and conclude that learning can contribute even more to the model’s explanatory power when its parameterization is more flexible. When applied to technology specific to housing market, the likeliest unrestricted specification of the learning mechanism is such that the economic agents largely disregard the possibility that house price dynamics can be subject to persistent changes. In case of the crisis of 2007–2009, such learning mechanism implies that housing market participants were stubbornly optimistic about the future of the market, which was why house prices were so slow to decline and the foreclosure rate was persistently high.
Chapter 2

LEARNING AND THE MARKET FOR HOUSING: A LIKELIHOOD ESTIMATION

2.1 Introduction

Chapter 1 presents two versions of a dynamic stochastic general-equilibrium model with endogenous housing market. In one version, economic agents have perfect knowledge about the persistence of exogenous shocks hitting the economy; the other version introduces imperfect knowledge and learning. A preliminary analysis of the dynamic features of the two models finds that the model with learning has a potential to better explain the sluggish dynamics in the market for housing in the U.S. The key intuition is that in presence of imperfect knowledge about the length of business cycles, agents only gradually learn about a long recession or boom. Therefore, persistent shocks do not create immediate adjustments to observable variables; instead, the response to such shocks is protracted. In a DSGE model, learning creates inertia observed in the U.S. market for housing.

For a proper formal test of this claim, a Bayesian likelihood-based analysis seems to be in order. The results of a maximum-likelihood estimation performed in Chapter 1 correspond to two different structural models rather than different parameter specifications within one structure. The parameter vector is transformed into the state-space form differently for the two models, so the maximized likelihood functions are not directly comparable. Bayesian estimation, on the contrary, helps compare the two models by means of posterior odds. An additional methodological benefit that comes with Bayesian inference is that it does not require an assumption (null hypothesis) that one of the DSGE models is true; the two models are treated symmetrically.\footnote{See Fernández-Villaverde and Francisco Rubio-Ramírez (2004) for a discussion of advantages of posterior-odds ratio.}

The goal of this chapter is to conclude which of the two competing DSGE models
from Chapter 1 is a likelier explanation of the key U.S. data on consumption, investment, and housing, which spans 1975–2013. I form a prior about the distribution of the key parameters that define the dynamic characteristics of the two models. Then, I evaluate both models against the data and find the parameters’ posterior distributions by means of a Markov chain Monte Carlo integration with Metropolis random-walk sampling algorithm (An and Schorfheide, 2007). With a few exceptions, the results are in line with MLE output presented in Chapter 1. Finally, I use the routine described by Chib and Jeliazkov (2001) to find the marginal likelihoods and compute the posterior-odds ratio for the two models. The resulting odds are 90:1 in favor of the model with imperfect knowledge and learning, a strong evidence. To explain how the model with learning is better at capturing the dynamics of the observed variables, I provide variance decomposition for the two models.

In this chapter, section 2.2 outlines the method to deliver the empirical results; section 2.3 describes the results; section 2.4 looks into variance decomposition; and section 2.5 concludes.

2.2 Methodology

Let \( P \) denote the model with perfect knowledge, and \( I \) denote the model with imperfect knowledge about shock persistence. In order to compare the empirical performance of the two models, I use posterior odds in favor of the model \( I \):

\[
PO = \frac{p(Z|I) \pi(I)}{p(Z|P) \pi(P)}.
\]

The first fraction, the Bayes Factor, contains marginal likelihoods that stand for probabilities of the observed data \( Z \) given a specified model \( M \in \{I, P\} \). The second fraction contains the prior belief about the likelihood of the two models. I assume no prior odds, so the ratio is reduced to the Bayes Factor:

\[
PO = \frac{p(Z|I)}{p(Z|P)}. \tag{2.1}
\]
Marginal likelihood, therefore, is the central goal of the outlined empirical exercise:

\[ p(Z|M) = \int L(Z|\theta, M)\pi(\theta|M)d\theta. \] (2.2)

Here, \( L(Z|\theta, M) \) is the likelihood function evaluated for a parameter vector \( \theta \), and \( \pi(\theta|M) \) is its prior density. The deep parameters that define the structure of the model are grouped into two vectors, \( \theta_C \) and \( \theta \). Vector \( \theta_C \) contains parameters that are calibrated to match the steady state of the model to certain empirical targets. I treat vector \( \theta_C \) as fixed and focus the Bayesian inference on vector \( \theta \), which contains the parameters that are most responsible for the dynamic behavior of the model: AR coefficients \( \{\rho_i\}_i \) and standard deviations \( \{\sigma_j\}_j \) of exogenous shock processes, as well as technological growth rates \( \{\gamma_k\}_k \).

Given a parameter specification \( \theta \), I can cast either one of the models in linear state-space form and apply Kalman filter in order to compute the value of the likelihood function.\(^3\) Following the literature, I specify a tractable prior parameter distribution \( \pi(\theta|M) \). However, I do not have a tractable posterior distribution of \( \theta \), which equals the product of the two terms (up to a constant factor of proportionality): \( p(\theta|M, Z) \propto L(Z|\theta, M)\pi(\theta|M) \). Therefore, I cannot find the marginal likelihood analytically. Moreover, direct sampling from the posterior distribution is impossible, which complicates numerical integration of the expression (2.2).

To overcome this problem, I use Markov chain Monte Carlo (MCMC) sampling with Metropolis random-walk algorithm, which is the common approach in the literature.\(^4\) In brief, the sampling proceeds as follows. Given current draw \( \theta_k \), a candidate to become the next draw \( \theta^*_{k+1} \) is sampled from a well-defined distribution \( q(\theta|\theta_k) \), which serves to replace the unknown posterior distribution \( p(\theta|M, Z) \). The candidate \( \theta^*_{k+1} \) is accepted as the next

---

\(^2\)Notice that I have to de-trend the data every time I compute the likelihood \( L(Z|\theta, M) \) for a certain \( \theta \).

\(^3\)Chapter 1 describes the procedure that converts both competing models into linear Gaussian state-space forms, by means of de-trending and log-linearization. See section 1.2 for details.

draw $\theta_{k+1}$ with probability $\alpha(\theta_k, \theta^*_k)$:

$$
\alpha(\theta_k, \theta^*_k) = \min \left\{ \frac{L(Z|\theta^*_{k+1}, \mathcal{M}) \pi(\theta^*_{k+1}|\mathcal{M}) q(\theta^*_{k+1}|\theta_k)}{L(Z|\theta_k, \mathcal{M}) \pi(\theta_k|\mathcal{M}) q(\theta_k|\theta^*_{k+1})}, 1 \right\}.
$$

(2.3)

With probability $1 - \alpha(\theta_k, \theta^*_k)$, the candidate is rejected, and the current observation is accepted as the new draw: $\theta_{k+1} = \theta_k$.

A proper distribution $q(\theta'|\theta)$ should resemble the posterior density $p(\theta|\mathcal{M}, Z)$ and have adequately thick tails, so that the sampling algorithm vigorously traverses the non-trivial part of the domain of the posterior density. I choose a normal distribution: $q(\theta'|\theta) \sim N(\theta, (X'X) \times \Sigma)$. To obtain a suitable variance-covariance matrix $\Sigma$, I use numerical hill-climbing methods to find $\hat{\theta}$ that maximizes the function $g(\theta) = L(Z|\theta, \mathcal{M}) \pi(\theta|\mathcal{M})$ and employ the inverse of its Hessian evaluated at $\hat{\theta}$. The term $(X'X)$ represents a tuning factor that modifies the estimated matrix $\Sigma$ in order for the sampling algorithm to accept the new candidates at a suitable rate. I use a scalar in lieu of this factor, such that the acceptance rate is between 0.25 and 0.35. Because $q(\theta'|\theta)$ is symmetric, the fraction $q(\theta^*_{k+1}|\theta_k)/q(\theta_k|\theta^*_{k+1})$ can be dropped from expression (2.3). I set the sample size at $M = 400,000$ to ensure convergence. I choose the initial draw $\theta_0$ randomly in the vicinity of the estimated posterior mode $\hat{\theta}$, and run a burn-in stage of 50,000 draws. To reduce autocorrelation, I thin the series down to every 100th draw. Section 2.3.3 discusses quality of the obtained samples.

Given a proper choice of the stand-in distribution $q(\theta'|\theta)$, a starting point $\theta_0$, and a sufficiently large thinning step, the described technique replicates random sampling from the posterior distribution $p(\theta|\mathcal{M}, Z)$. Therefore, I can estimate the moments of the posterior distribution using their sample counterparts. The marginal likelihood $p(Z|\mathcal{M})$, however, cannot be represented as a moment of the posterior distribution.\(^5\) Chib and Jeliazkov (2001) provide an efficient approximation to marginal likelihood that employs the output

\(^5\)See Geweke (1999) for details.
of MCMC sampling. The approach is based on the basic marginal likelihood identity:

\[
p(Z|M) = \frac{L(Z|\theta^*_M, M) \pi(\theta^*_M|M)}{p(\theta^*_M|M, Z)}, \tag{2.4}
\]

where the likelihood function, the prior density, and the posterior ordinate are evaluated at a high-density point \(\theta^*_M\) of the posterior distribution, such as the posterior mode \(\hat{\theta}_M\) estimated for each model. The only unknown term on the right-hand side of the identity is the posterior ordinate, which is estimated as

\[
\hat{p}(\theta^*_M|Z, M) = \frac{M^{-1} \sum_{i=1}^{M} \alpha(\theta_k, \theta^*_M) q(\theta^*_M|\theta_k)}{J^{-1} \sum_{j=1}^{J} \alpha(\theta^*_M, \theta_j)}, \tag{2.5}
\]

where \(\{\theta_k\}_{k=1}^M\) are draws simulated by the MCMC algorithm, and \(\{\theta_j\}_{j=1}^J\) are a separate sample drawn from \(q(\theta|\theta^*_M)\).

The following section provides the results of the described procedures.

2.3 Estimation

2.3.1 Calibration and Prior Distributions

Likelihood-based estimation procedures that I use focus on log-deviations from the steady state of a de-trended model. For this reason, deep parameters that mostly define the steady state are difficult to estimate within this framework.\(^6\) I calibrate such parameters to make the steady state of the model match key empirical targets that are true for the U.S. housing market (e.g., mortgage default rate, loan-to-value ratio, etc.) and for the economy in general (e.g., the great ratios \(I/Y, K/Y\)). The calibration is the same for both estimated models, since imperfect knowledge and learning are purely dynamic characteristics of the model that do not affect the steady state.\(^7\)

As for the likelihood-based estimation, it targets the parameters that define the trend

\(^6\)For example, Iacoviello and Pavan (2012) and Leeper et al. (2010) provide similar arguments.

\(^7\)Section 1.3.2 of chapter 1 describes the procedure and table 1.1 summarizes the calibrated values.
and the features of the shock processes. To form their prior, I use the existing literature, as well as the results of MLE reported in Chapter 1. In particular, these results show that technological growth rates fall within two percentage points from zero. Their prior is specified as normal distribution with mean 0.001 and standard deviation 0.01. Standard deviations of all the shock processes are assumed to have inverse gamma distribution with mean 0.001 and standard deviation 0.01. Such prior allows for consideration of near-zero standard deviations. All the auto-regressive coefficients are assumed to have beta distribution with mean 0.7 and standard deviation 0.15 in the prior, which is loose enough to allow a wide range of estimates. Overall, this prior is similar to the one used by Iacoviello and Pavan (2012). Table 2.1 summarizes the prior and figures 2.1 and 2.2 illustrate it graphically.

2.3.2 Posterior Distributions and Posterior Odds

Table 2.1 reports the key moments of the posterior distributions estimated for the two models, and figures 2.1 and 2.2 compare them to prior distributions. They are generally in line with Chapter 1: parameter estimates from Chapter 1 (see table 1.3) are within the 95%-confidence intervals of the posterior distributions. The posterior distributions estimated for the model with imperfect knowledge ($I$) are significantly different in a few notable cases, however. The growth rate of construction technology $\gamma_x$ seems to be distributed much lower than the value $\gamma_x^{MLE} = -0.0032$ reported in Chapter 1. It translates into lower housing stock production and a higher long-term growth rate of house price generated by model $I$. In addition, two processes, persistent component of consumption technology growth and retained housing shock, show significantly higher AR coefficients ($\rho_y$ and $\rho_\Omega$) and lower standard deviations ($\sigma_{vy}$ and $\sigma_\Omega$). This could be due to impact of prior distributions that are focused above 0.5 for AR coefficients and close to zero for standard deviations.

The key differences between the estimates for the two models remain the same as in Chapter 1. It is only in case of imperfect knowledge that technologies of consumption-good and capital creation have both persistent and transitory components with sizable variances.
Table 2.1: Prior and posterior distributions for the two estimated models

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Prior distribution</th>
<th>Posterior, model $P^*$</th>
<th>Posterior, model $I^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Type</td>
<td>Mean</td>
<td>St. D.</td>
</tr>
<tr>
<td>$\gamma_y$ Consumption tech. growth</td>
<td>$N$</td>
<td>0.005</td>
<td>0.01</td>
</tr>
<tr>
<td>$\gamma_x$ Construction tech. growth</td>
<td>$N$</td>
<td>0.005</td>
<td>0.01</td>
</tr>
<tr>
<td>$\gamma_k$ Capital tech. growth</td>
<td>$N$</td>
<td>0.005</td>
<td>0.01</td>
</tr>
<tr>
<td>$\rho_y$ Consumption tech. AR coeff.</td>
<td>$B$</td>
<td>0.700</td>
<td>0.15</td>
</tr>
<tr>
<td>$\rho_x$ Construction tech. AR coeff.</td>
<td>$B$</td>
<td>0.700</td>
<td>0.15</td>
</tr>
<tr>
<td>$\rho_k$ Capital tech. AR coeff.</td>
<td>$B$</td>
<td>0.700</td>
<td>0.15</td>
</tr>
<tr>
<td>$\rho_\psi$ Housing preference AR coeff.</td>
<td>$B$</td>
<td>0.700</td>
<td>0.15</td>
</tr>
<tr>
<td>$\rho_\nu$ Intertemp. preference AR coeff.</td>
<td>$B$</td>
<td>0.700</td>
<td>0.15</td>
</tr>
<tr>
<td>$\rho_\Omega$ Retained housing AR coeff.</td>
<td>$B$</td>
<td>0.700</td>
<td>0.15</td>
</tr>
<tr>
<td>$\sigma_{vy}$ Consumption pers. st. dev.</td>
<td>$G^{-1}$</td>
<td>0.001</td>
<td>0.01</td>
</tr>
<tr>
<td>$\sigma_{vx}$ Construction pers. st. dev.</td>
<td>$G^{-1}$</td>
<td>0.001</td>
<td>0.01</td>
</tr>
<tr>
<td>$\sigma_{vk}$ Capital pers. st. dev.</td>
<td>$G^{-1}$</td>
<td>0.001</td>
<td>0.01</td>
</tr>
<tr>
<td>$\sigma_{uy}$ Consumption trans. st. dev.</td>
<td>$G^{-1}$</td>
<td>0.001</td>
<td>0.01</td>
</tr>
<tr>
<td>$\sigma_{ux}$ Construction trans. st. dev.</td>
<td>$G^{-1}$</td>
<td>0.001</td>
<td>0.01</td>
</tr>
<tr>
<td>$\sigma_{uk}$ Capital trans. st. dev.</td>
<td>$G^{-1}$</td>
<td>0.001</td>
<td>0.01</td>
</tr>
<tr>
<td>$\sigma_\psi$ Housing preference st. dev.</td>
<td>$G^{-1}$</td>
<td>0.001</td>
<td>0.01</td>
</tr>
<tr>
<td>$\sigma_\nu$ Intertemp. preference st. dev.</td>
<td>$G^{-1}$</td>
<td>0.001</td>
<td>0.01</td>
</tr>
<tr>
<td>$\sigma_\Omega$ Retained housing st. dev.</td>
<td>$G^{-1}$</td>
<td>0.001</td>
<td>0.01</td>
</tr>
</tbody>
</table>

$^*$ $P$ denotes the model with perfect knowledge; $I$ denotes the model with imperfect knowledge.
As impulse-response functions have indicated, the model with imperfect knowledge can use these exogenous processes effectively to explain the dynamics of the observable variables. First, in presence of significant transitory shocks, learning allows the persistent shocks to have lasting impact on the model’s state, while keeping the immediate response subdued—this feature is particularly useful for generating low-frequency dynamics. And second, learning can wash out bad co-movement immediately upon a persistent shock, while keeping its long-term effect. The latter is especially applicable to capital-technology shocks. Because model $\mathcal{P}$ cannot employ the pairs of persistent and transitory technology shocks as effectively, it relies on other processes to interpret the data. Since capital technology is estimated to have infinitesimal variances in case of perfect knowledge, inter-temporal preference shock process replaces it to explain investment dynamics. The standard deviation $\sigma_\nu$ seems to be very large for model $\mathcal{P}$, with posterior mean at 0.0472 and the 95%-confidence interval spanning values up to 0.1019. These high values put the validity of linear approximation of the model under question.

The idea that the model with imperfect knowledge is better at mimicking the sluggish dynamics of the observable variables finds its formal confirmation. I estimate the marginal likelihoods of the two models and find the following key result:

$$\frac{\hat{p}(Z|I)}{\hat{p}(Z|\mathcal{P})} \approx 90. \quad (2.6)$$

The odds are estimated to be 90:1 in favor of the model with imperfect knowledge, which, according to Dave and DeJong (2007), signifies "strong evidence". In order to better explain why the model $\mathcal{I}$ uses combinations of transitory and persistent shocks more efficiently, I analyze variance decomposition for the two models in section 2.4 below. Before any further analysis, a note on quality of the performed simulations is in order.
Figure 2.1: Posterior distributions: perfect knowledge. Black (dashed) lines represent the prior distributions; red (solid) lines represent the posterior distributions.
Figure 2.2: Posterior distributions: imperfect knowledge. Black (dashed) lines represent the prior distributions; red (solid) lines represent the posterior distributions.
2.3.3 Quality of Simulated Samples

Figures 2.1 and 2.2 indicate that the data set is uninformative with respect to certain parameters; for these parameters, the estimated posterior distribution is similar to the prior. In case of model $\mathcal{P}$, it is true for capital-creation technology, transitory component of consumption-sector technology, and housing-preference shock. In case of model $\mathcal{I}$, it applies to housing-preference and retained-housing shock processes. The reason is that these shock processes do not add explanatory power to the models. Why? Suppose that, for a model $\mathcal{M}$, a shock process $s$ is unimportant, so that the true value of its standard deviation $\sigma_s$ is zero. Then, the likelihood function $L(Z|\theta, \mathcal{M})$ will achieve high values when the standard deviation $\sigma_s \in \theta$ is chosen close to zero. Therefore, within the Metropolis sampling algorithm, a parameter candidate $\theta^{*+}_{k+1}$ has a chance to become an accepted draw if it is such that $\sigma^{*}_{s,k+1} \approx 0$, where $\sigma^{*}_{s,k+1} \in \theta^{*}_{k+1}$. The prior for standard deviations is such that most of its density is concentrated in the region below 0.002. If the standard deviation $\sigma^{*}_{s,k+1}$ of an 'unimportant' shock process is sampled from this region, its value is too small for any significant negative impact on the likelihood. The sampling routine will deliver the posterior that will be almost identical to the prior. In addition, given next-to-zero variance, the AR coefficient has no impact on the likelihood. Therefore, the posterior of the AR coefficient should be similar to the prior if the corresponding standard deviation is distributed very close to zero.

The cost of having such parameters in vector $\theta$ is lower quality of the sample simulated by the MH algorithm. When a series generated by the sampling algorithm is of high quality, its plot resembles a white-noise process. For this to happen, the variances of the sampling distribution $q(\theta'|\theta)$ must be comparable in size to the variances of the posterior distributions. This is problematic for the parameters describing the 'unimportant' shock processes, for which the posteriors are similar to the prior distributions. A sampling distribution with large variances for such parameters would often supply candidates $\theta^{*}_{k+1}$
that contain unrealistic standard deviations or AR coefficients ($\sigma^* < 0$ or $\rho^* \notin [0, 1]$). In order to maintain the acceptance rate of the sampling algorithm, the sampling distribution $q(\theta'|\theta)$ must assign low variances to the troubled parameters. The result is that the series simulated for these parameters exhibit poor mixing, high serial correlation even after thinning, and slow convergence of the sample moments.

In part B of the appendix, figures B.1–B.6 plot the series along with their autocorrelations and recursive means. These measures indicate lower quality of the sample for the parameters of the 'unimportant' shock processes. A straightforward way to improve the quality of sampling for these parameters is to increase the sample size and the thinning step. This is quite time-consuming. It seems sufficient, however, to notice that the estimated posterior distribution $\hat{p}(\theta|\mathcal{M}, Z)$ shows no significant covariance between individual parameters. Therefore, the problem with quality of sampling is not contagious.

As long as standard deviations of the 'unimportant' shock processes are drawn from the vicinity of zero, their values, along with the corresponding AR coefficients, seem to have little impact on the likelihood $L(Z|\theta^*_k+1, \mathcal{M})$, on the validity of the results for the rest of the parameters, and marginal likelihood in general.

### 2.4 Interpreting the Results: Variance Decomposition

#### 2.4.1 Forecast variance

Figure 2.3 demonstrates forecast variance decomposition of the observable variables for the two models; it helps better understand the relative importance of shocks. Consumption technology accounts for most of variability in consumption; virtually all of it in the long run. Consumption and capital technologies explain most of variability in capital investment, since capital investment depends on efficiency of capital production and on

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8Notice that the prior density for AR coefficients is loose yet bounded by zero and one; while the prior density for standard deviations is skewed to the right and cut off by zero from below.

9Another indication of independence of the parameters in the posterior is that the inverse of the Hessian, which is estimated at the maximum $\hat{\theta}$ of $g(\theta) = L(Z|\theta, \mathcal{M})\pi(\theta|\mathcal{M})$ and used as a variance-covariance matrix $\Sigma$ in the sampling distribution $q(\theta'|\theta)$, is practically a diagonal matrix.
Figure 2.3: Forecast variance decomposition for the two models. Bar heights represent percentages of the variance of the forecast growth contributed by each of the shocks. The forecast horizon is up to 40 quarters.

availability of consumption good to spend on capital. In case of perfect knowledge, there is no capital technology shock; instead, there is inter-temporal preference shock that affects the consumption-saving decision by households and accounts for a lot of variation in capital investment, even for longer horizon (around 25% for a 40-quarter forecast). Variability in residential investment and house price is mostly due to construction technology and housing supply shocks. Notably, housing supply shock, despite its transitory nature, remains an important contributor to the variance of residential investment (up to 30–60 percent) even
for longer horizons—this is because retained housing is one of the inputs of construction sector, and a decrease in housing stock suppresses construction for a long time. For both capital investment and consumption, housing supply shock is important in the short run because, through house prices, it affects the share of wealth households allocate to housing purchases rather than consumption and saving. Around 20% of house-price variance is also due to consumption technology shocks, since houses are paid for in units of consumption good.

Technology shocks seem to be the most important drivers; and more so over the longer forecast horizon, since they create permanent deviations from the balanced growth path. Persistent technology shocks, despite their small estimated variances, are especially important. Under imperfect knowledge, persistent components of capital and consumption technologies prove to be important for all four variables. Under perfect knowledge, persistent component of construction technology matters for capital and residential investment and especially for house price. The only consumption technology component that exists under perfect knowledge is the persistent one, which contributes a lot to all the observable variables but residential investment. The conclusion is that, to a large degree, long-run evolution of variables simulated by the two models can be attributed to technology shocks, and in particular to their persistent components.

2.4.2 Historical variance

Figures 2.4 and 2.5 depict the historical variance decomposition of growth in observable variables according to the model with imperfect knowledge. The two figures clarify the nature of technology shocks and the ways they explain the data. Technology shocks are important for low-frequency dynamics; they have permanent impact on the balanced growth path, and they build up their influence on the course of economy over time. For these reasons, technology shocks, especially persistent shocks, contribute the most during the periods of long-lasting steady growth or decline in the variables; and the abrupt swings in growth are likely to be due to transitory technology or non-technology shocks. This is in line with
Figure 2.4: Historical variance decomposition, imperfect knowledge, aggregate consumption and capital investment. White dots represent the observed percentage growth in the variables; bar heights represent contributions of each of the shock to the growth.
Figure 2.5: Historical variance decomposition, imperfect knowledge, residential investment and house price. White dots represent the observed percentage growth in the variables; bar heights represent contributions of each of the shock to the growth.
the common intuition that technology processes drive the economy in the long run, while the short-run fluctuations could be also due to a plethora of transitory factors, such as preference shocks, financial shocks, monetary policy shocks, etc.

Persistent components of consumption- and capital-specific technologies are important for variation in aggregate consumption and investment. Figure 2.5 shows that house price growth also depends on persistent consumption-specific technology shock; however, its contribution is very often in the opposite direction to the observed growth. In other words, the general direction of house price is not always in line with consumption; there are other factors that drive the house price growth. A booming aggregate economy and a growing aggregate consumption do not always correspond to growing house prices, and vice versa. This observation may point to the existence of house price bubbles; alternatively, it may point to economic developments specific to housing market, such as introduction and spread of mortgage-backed securities. The decomposition shows that variables specific to housing market are the ones that define the evolution of house price. Housing supply shock accounts for a lot of abrupt swings of house price, and construction technology defines the direction of house price movement that generally prevails over several quarters.

However, the model with imperfect knowledge renders the persistent component of construction technology insignificant, while it could be the source of more persistent dynamics of house price, like it is in the case of perfect knowledge.\(^{10}\) The implication is that the model with imperfect knowledge may not generate much additional inertia in house price compared to the model with perfect knowledge. As I argue in section 1.4.3 of Chapter 1, a way to address this problem is to consider more flexible learning processes that are not necessarily based on the true features of technology processes. This is an interesting direction to take this empirical work in the future.

\(^{10}\)Appendix B provides historical variance decomposition for the case of perfect knowledge.
2.5 Conclusion

Likelihood-based Bayesian inference is the appropriate strategy to compare different structural models that compete to describe the observed data. I apply this commonly used technique to the two models introduced in Chapter 1. I find that imperfect knowledge and learning about the persistence of business cycles significantly improve the empirical performance of a DSGE model with housing. The estimated odds are 90:1 in favor of the model that assumes imperfect knowledge, which is strong evidence. Variance decomposition suggests that, because the observed data exhibits inertia, the model with learning is a better fit due to its ability to use persistent shocks to generate low-frequency dynamics.
Chapter 3

SOVEREIGN DEBT MATURITY IN EMERGING ECONOMIES

3.1 Introduction

Despite the obvious differences, unsecured government debt is theoretically similar in many ways to household debt; and, like the household debt, it has been a central topic during the Great Recession. Debt maturity is an important aspect of business cycles, since it affects the ability of the government to service the debt and efficiently reallocate resources between different moments in time. A sovereign with a large short-term debt has to rely on subsequent rollovers, which may be costly in bad times and even trigger a sovereign debt crisis or outright default.\footnote{For an example, see Cole and Kehoe (2000).}

Why do emerging economies tend to borrow short-term during recessions, then? Recent literature has provided evidence as well as explanations for this phenomenon. Broner et al. (2013) suggest that optimal maturity is an outcome of a risk-sharing problem: the borrower prefers long-term debt because it provides more flexibility at arranging for debt repayment and protects against rollover crises when the prospects of the economy deteriorate; the lenders prefer short-term debt because it is less exposed to price risk. The authors argue that investors become more risk-averse during recessions and the equilibrium debt maturity shortens. In other words, a shorter maturity is a result of a shock to the supply of loanable funds. According to Arellano and Ramanarayanan (2012), the long-term debt insures the borrower against losses in cases when national income falls: a low-income state implies a higher chance of default and a lower value of outstanding debt. At the same time, investors can price the short-term debt more effectively in order to ensure its repayment; therefore, short-term debt has a higher price and provides a better access to liquidity.\footnote{Chatterjee and Eyigungor (2012) also argue that investors pay more for short-term bonds because the long-term debt suffers from lack of commitment to repay the debt.} During low-income states, it follows, the urgent need for liquidity on the borrower’s side makes short-term debt preferable.

1\footnote{For an example, see Cole and Kehoe (2000).}

2\footnote{Chatterjee and Eyigungor (2012) also argue that investors pay more for short-term bonds because the long-term debt suffers from lack of commitment to repay the debt.}
I propose a different explanation for shortened maturity during recessions in emerging economies. Consider an economy that is subject to income shocks of different persistence and its government looking to borrow short or long term. A persistent negative income shock may eventually lead to a default, whereas a transitory shock is likelier to end with a quick recovery. Immediately upon shock, investors are uncertain about the exact scenario they see unfolding. What they are certain about is that this uncertainty will rapidly resolve in favor of a persistent recession with low bond prices due to likely default or a transitory recession with high bond prices due to quick recovery and the sovereign likely to remain solvent. Therefore, uncertainty about the length of the recession makes long-term bonds exhibit larger price risk, so that investors shift towards short-term debt which matures before a lot of the uncertainty is resolved. In line with Broner et al. (2013), I argue that emerging economies tend to borrow short-term during recessions because long-term borrowing becomes relatively expensive. I attribute maturity-shortening to changes in supply of loanable funds as well. However, my explanation of pro-cyclical debt maturity does not rely on counter-cyclical investor risk-aversion; instead, the always-present uncertainty about the persistence of income fluctuations and learning about it adds price risk to long-term bonds and makes them costly to borrowers during recessions. Note that the cost of borrowing stands for yield spread net of probability of default, or the risk premium that the borrower expects to pay on debt.

The explanation that I provide, while intuitive, is also empirically plausible. Aguiar and Gopinath (2004) document that emerging economies exhibit income dynamics with a significant stochastic trend component. In a different work (Aguiar and Gopinath, 2006), the authors establish that accounting for such component can improve the empirical performance of a dynamic stochastic model of a small open emerging economy with one-period debt and strategic defaults; it creates counter-cyclical current account and interest rates. Recessions generated by trend shocks are persistent and likely to lead to defaults; as a re-

\[\text{There is literature that attributes pro-cyclical debt maturity to changes on the demand side. For example, Missale and Blanchard (1991) explain shorter maturity as a vehicle for the borrower to commit against moral hazard attributable to long-term debt, especially during times of high debt-to-GDP ratio.}\]
sult, borrowing becomes more expensive (bond price falls), so the current account increases. I argue that one more salient feature of a dynamic model with stochastic trend component is that, when coupled with imperfect information and learning about the source of income fluctuations, it can potentially generate empirically plausible pro-cyclical debt maturity. An interesting implication for the borrowing developed economies is that they should have a less pro-cyclical sovereign debt maturity because their national incomes exhibit relatively small trend components (Aguiar and Gopinath, 2004). I discuss this implication in section 3.3.

The goal of this work is to establish the key factors that shorten the equilibrium debt maturity at the onset of a recession and show that uncertainty about the persistence of the recession is one of these factors. To achieve this goal, I develop a simple three-period model of an emerging economy that is born in a recession of uncertain length with a large debt to restructure. In line with Broner et al. (2013), the model shows that equilibrium debt maturity is the outcome of risk-sharing between a policymaker that seeks to smooth the economy’s consumption path and risk-averse investors that purchase short- and long-term bonds and price them according to the expected payments they deliver in different states of the world upon maturity. A higher investors’ risk-aversion or a lower borrower’s risk-aversion result in a shorter debt maturity. Most importantly, the model is able to demonstrate that uncertainty about the type of recession shortens the debt maturity in equilibrium, compared to the case of no uncertainty. In addition, when there is uncertainty, if the long-recession scenario corresponds to a more persistent income shock, or if this scenario is believed to be likely, the debt maturity is shorter in equilibrium. Finally, the model shows that a high initial debt (that is, likely default) at the onset of a recession is required for the maturity to shorten.

In what follows, section 3.2 describes the model; section 3.3 delivers the results and develops the intuition; section 3.4 discusses applications and concludes.
3.2 Model

Consider an economy that generates a random national income available for consumption in every period. The income can be either high or low, and a benevolent policymaker, the risk-averse government, has a concave utility function over the aggregate consumption, and wishes to smooth consumption path. To do so, the government can borrow from risk-averse international investors. By issuing $D_i$ bonds, the government promises to pay $D_i$ units of consumption good in the period when these bonds mature. The economy is born in period 0 with a low income and an outstanding debt $D_0$ payable immediately; it lives for three periods and dies after period 2. In order to restructure its debt in period 0, the government can borrow $D_{L0}$ long-term bonds that mature in period 2 and $D_{S0}$ short-term bonds that mature in period 1. In period 1, the government cannot default; it has to repay the short-term debt $D_{S0}$, and it can issue $D_{S1}$ short-term bonds that mature in period 2. In period 2, the government can either repay the outstanding debt $D_{S1} + D_{L0}$ and consume the remaining income, or default on the debt and pay a fraction $\mu$ of national income as a cost of default. The risk-averse investors price the bonds consistently with their stochastic discount factor, the policymaker’s optimal decision to default in the last period, and the likelihood of state of default.

3.2.1 Income Process

The income process replicates the onset of a recession of uncertain persistence. Agents know that income is a two-state Markov chain: $y_t \in \{y_L, y_H\}$. The economy is born in period 0 with low income: $y_0 = y_L < y_H$. It is not known what transition matrix $P_i$ defines the income dynamics. In particular, agents believe that with probability $p$, income is a persistent process with transition matrix $P_1$ and with probability $1 - p$, income is a volatile
process with transition matrix $P_2$, such that

$$P_1 = \begin{pmatrix} 1 - s/2 & s/2 \\ s/2 & 1 - s/2 \end{pmatrix}; \quad P_2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix},$$

where $s \in [0,1)$. Let $l_0$ be the probability (estimated at period 0) that income in period 1 is low:

$$l_0 = \Pr_0(y_1 = y_L) = \frac{1 + p - sp}{2}.$$  \hfill (3.1)

Naturally, $l_0$ is higher for higher $p$ or lower $s$. Agents will use the information about the state of income in period 1 to learn about the likelihood of each of the transition matrices that compete to describe the income process. Low income in period 1 will make agents update their belief in favor of the persistent income process, and high income will make the transitory income process a likelier scenario. The probability of low income in period 2 estimated in period 1, therefore, depends on the state of income. For each of the two states of income in period 1, it is computed using the Bayes rule:

$$l_L = \Pr_1(y_2 = y_L|y_1 = y_L) = \frac{1}{2} \frac{p(2 - s)^2 + 1 - p}{1 + p - sp}; \hfill (3.2)$$

$$l_H = \Pr_1(y_2 = y_L|y_1 = y_H) = \frac{1}{2} \frac{ps^2 + 1 - p}{1 - p + sp}. \hfill (3.3)$$

The diagram below summarizes the possible paths of income process and the likelihood of each of the paths estimated by the agents.

\[\begin{array}{c}
\bullet & y_H & y_H \\
1 - l_0 & \bullet & \bullet \\
l_0 & \bullet & \bullet \\
\bullet & y_L & y_L \\
1 - l_L & l_L & \scriptsize{y_L, \text{ possible default}} \\
\end{array}\]
Note that forecast income dynamics between the two states are summarized by two parameters, \( p \) and \( s \). Parameter \( p \) defines the initial likelihood of the persistent Markov chain, and parameter \( s \) defines the degree to which it is persistent, with \( s \) close to zero standing for high persistence. This is a counterpart of an infinite-horizon set-up in which income is driven by two components, transitory and persistent shock processes, while agents can only observe the aggregate income and implement Kalman filtering to distinguish between the sources of income dynamics. In such set-up, the Kalman gain, or the degree to which the agents will be prone to interpret the changes in income as driven by the persistent component, depends on the volatility of the persistent component relative to the transitory one (signal-to-noise ratio) and on the persistence of the persistent component. For example, when the income is hit by a negative shock, the former tells how likely the shock is to be persistent immediately upon shock, and the latter tells how quickly the agents will learn that the shock is persistent if the income stays low. Parameters \( p \) and \( s \) in my model work in exactly the same manner.

### 3.2.2 Bond Prices

The government can issue bonds and sell them to risk-averse investors. At period 0, the government can sell both short- and long-term bonds, which mature in periods 1 and 2, respectively. At period 1, the government can only issue short-term, one-period bonds. For simplicity, assume the government cannot save or buy back the long-term bonds in period 1.\(^4\) Also, as it is explained below, default can occur in equilibrium only after the low-income state in period 1. Therefore, credit risk is an issue only when the income remains low in period 1, which is one reason why the long-term bonds are subject to price risk. Another reason is that bond prices depend on the behavior of the stochastic discount

\(^4\)Broner et al. (2013) report that borrowing governments almost never buy back the long-term bonds. Saving in period 0 is not optimal due to a large outstanding debt. Saving in period 1 is restricted for simplicity; this restriction partially impedes optimal consumption-smoothing, but does not distort the qualitative results of the model—see section 3.3.
factor, or pricing kernel, $m_t$ attributable to investors:

$$m_t = \exp(-\alpha \ln y_t - \gamma \ln x_t + c),$$

(3.4)

where $y_t$ is the economy’s income, $c$ is a constant, and $x_t$ is a log-normally distributed random variable: $\ln x_t \sim N(0, \sigma^2)$.\(^5\) Parameter $\alpha$ defines the degree to which the pricing kernel $m_t$ is correlated with the economy’s income. Assumably, the markets for public debt are segmented, and investors are specialized in given economy so that a significant portion of their wealth is invested in the economy and not diversified. Another reason for correlation could be that the economy’s income can be subject to systemic shocks, such as global business cycles.\(^6\) The unconditional expectation of the pricing kernel is the following:

$$E[m] = E[y^{-\alpha}]E[e^{-\gamma \ln x + c}] = \left(\bar{l}y_L^{-\alpha} + (1 - \bar{l})y_H^{-\alpha}\right) e^{\gamma^2 \sigma^2 / 2 + c},$$

where $\bar{l}$ is the unconditional probability of low-income state. Given that both transition matrices $P_1$ and $P_2$ are symmetric, it is safe to say that $\bar{l} = 1/2$. Parameter $c$ is calibrated so that the unconditional expectation of the pricing kernel corresponds to a constant risk-free rate $r$, which I calibrate and which corresponds to a risk-free asset unavailable to investors:

$$c = -\gamma^2 \sigma^2 / 2 - \ln \left(\left(y_L^{-\alpha} + y_H^{-\alpha}\right)/2\right) - \ln(1 + r) \implies E[m] = \frac{1}{1 + r}.$$

Given the pricing kernel, standard derivations based on the Euler equation for investors imply that a zero-coupon, short-term (or one-period) bond that promises one unit of con-

\(^5\)Arellano and Ramanarayanan (2012) discuss a similar kernel.

\(^6\)In an infinite-horizon dynamic framework, the latter explanation is less favored, since incomes in emerging economies show substantial trend components, unlike in developed economies. Therefore, the latter explanation is less favored, since income fluctuations should not be synchronized across different types of economies.
sumption good in period \( t + 1 \) has the following price in period \( t \):

\[
q_{St} = E_t[m_{t+1}\chi_{t+1}],
\]

where \( \chi_t \) is the indicator function that equals one whenever the government decides to repay the debt and zero whenever the government defaults. This simple formulation relies on the assumption that investors cannot recover any fraction of the government debt in case of default.\(^7\) Also, by construction, the government can only default in period 2. Therefore, \( \chi_1 = 1 \), and the one-period bond issued in period 0 has the following price:

\[
q_{S0} = \frac{1}{1 + r} \frac{l_0y_L^{-\alpha} + (1 - l_0)y_H^{-\alpha}}{l_0y_L^{-\alpha} + (1 - l)y_H^{-\alpha}}.
\]

(3.5)

The term \( l_0y_L^{-\alpha} + (1 - l_0)y_H^{-\alpha} \) stands for the unconditional expectation of income. Intuitively, because \( \alpha > 0 \), if the low-income state in the next period is likely, the bond price is higher. This is because the short-term bond promises a non-contingent payment in period 1, whereas investors are affected by the state of the economy (e.g., due to lack of diversification, they may suffer losses in wealth and consumption when the economy experiences a low aggregate income).

In period 1, the price of a short-term bond depends on the state of income and on the government’s policy which defines whether the default is likely. If the policy is such that default will never happen in period 2, the bond is called "safe" and its price is denoted by \( \tilde{q}_{Si} \), where \( i \in \{L, H\} \) denotes the state of income in period 1. If the government policy makes default in period 2 likely, the bonds are called "risky", and the first-period price of a short-term bond is labeled \( \tilde{q}_{Si}, i \in \{L, H\} \). The only feasible outcome of the model is that default, if likely at all, occurs only when the last-period income is low. The reason is provided in section 3.2.3 below. In short, excessive borrowing that makes default in the

\(^7\)Yue (2010) finds that a model with strategic defaults that includes endogenous debt renegotiation and partial debt recovery can have a better empirical performance (e.g., have higher equilibrium debt-to-GDP ratio and default rate).
last period certain is pointless because the corresponding bond price is zero; also, it is never optimal to prefer default in case of high income and repayment in case of low income in period 2.

It is intuitive to rewrite the equilibrium price of a short-term bond in period 1 using the covariance between the indicator for debt repayment and the pricing kernel:

\[ q_{Si} = E_i[m_2]E_i[\chi_2] + Cov_i(m_2, \chi_2), \quad i \in \{L, H\}. \]

Clearly, foreign lenders are willing to pay more for bonds if they expect the income to be low at the time of repayment, and less if the government is expected to default in low-income states. If the bonds are safe, the covariance term is zero, because \( \chi_2 = 1 \), and the price is

\[ \bar{q}_{Si} = \frac{1}{1 + r} \frac{ly_L^{-\alpha} + (1 - l_i)y_H^{-\alpha}}{ly_L^{-\alpha} + (1 - \bar{l})y_H^{-\alpha}}, \quad i \in \{L, H\}. \] (3.6)

If the bonds are risky, the indicator \( \chi_2 \) is perfectly correlated with income, and the price is

\[ \tilde{q}_{Si} = \frac{1}{1 + r} \frac{(1 - l_i)y_H^{-\alpha}}{ly_L^{-\alpha} + (1 - \bar{l})y_H^{-\alpha}}, \quad i \in \{L, H\}. \] (3.7)

The price of risky bonds reflects the value of the payment that investors will receive only in case of high income in period 2.

What about the price of the long-term bonds issued in period 0? The straightforward way is to combine equations (3.6) and (3.7) with the following recursion

\[ q_{L0} = E_0[m_1m_2\chi_2] = E_0[m_1q_{S1}]. \] (3.8)

Given the amount of borrowing \( \{D_{L0}, D_{S0}\} \) in period 0, lenders can predict the government policy for every state of the world in periods 1 and 2 and price the long-term bonds consistently with the policymaker’s optimal decision to default or repay in the terminal period.
3.2.3 Policymaker’s Problem

In each of the three periods $t \in \{0, 1, 2\}$, the policymaker acts benevolently to maximize the expected lifetime utility function derived from a representative agent’s consumption:

$$U_t = \sum_{i=0}^{2-t} \beta^i E_t [u(c_{t+i})].$$

(3.9)

I assume a concave instant utility function $u(c)$, so that the borrower is risk-averse. Let us consider the three periods consecutively.

3.2.3.1 Initial Period

The economy is born in period $t = 0$ with low income $y_L$ and a debt $D_0$ payable immediately. The government cannot default in period 0, so that $\chi_0 = 1$. It chooses the optimal amounts of long- and short-term debt $\{D_{L0}, D_{S0}\}$ in order to maximize the utility function (3.9). All the debt is restricted to be non-negative: $D_j \geq 0$. The value function in period 0 is the following:

$$V_0(y_L, D_0) = \max_{D_{L0}, D_{S0}} u(c_0) + \beta E_0[V_1(y_1, D_{L0}, D_{S0})].$$

(3.10)

The budget constraint in period 0 bounds consumption $c_0$ by national income net of debt payment $D_0$ plus the amount of funds provided by the foreign lenders for the newly issued bonds:

$$c_0 \leq y_L - D_0 + q_{S0}D_{S0} + q_{L0}D_{L0}.$$ 

(3.11)

Note that the government can commit to debt repayment in the future only if it is the optimal policy; lenders can predict the outcome for any debt combination $\{D_{L0}, D_{S0}\}$ in any future state of the world and set the prices $q_{L0}(D_{L0}, D_{S0})$ and $q_{S0}(D_{L0}, D_{S0})$ according

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8As mentioned above, it partially impedes consumption-smoothing in period 1, because saving in high-income state in period 1 may be optimal, but does not affect the qualitative results of the model. As for period 0, given the scope of outstanding debt $D_0$ and low income, it is optimal for the government to restructure the debt and remain a net borrower.
to equations (3.5)-(3.8).

### 3.2.3.2 Intermediate period

The economy enters period 1 with a debt combination \( D_{L0}, D_{S0} \), and the state of income \( y_1 \) is revealed. The agents update their beliefs about the probabilities of income states in the terminal period summarized by variable \( l_1 \in \{ l_L, l_H \} \), as described by equations (3.2) and (3.3). Default is not an option \( (\chi_1 = 1) \); the policymaker repays the short-term debt \( D_{S0} \) and issues an additional amount of short-term bonds \( D_{S1} \) that mature in the terminal period in order to maximize the utility function (3.9), which gives the following definition of the value function for period 1:

\[
V_1(y_1, D_{L0}, D_{S0}) = \max_{D_{S1}} u(c_1) + \beta E_1 [V_2(y_2, D_{L0}, D_{S1})].
\]  

The budget constraint is similar to (3.11):

\[
c_1 \leq y_1 - D_{S0} + q_{S1} D_{S1}.
\]  

Again, the bond price \( q_{S1} \) defined by equations (3.6) and (3.7) is consistent with the policymaker’s optimal decision to default in period 2 given the outstanding debt \( D_{L0} + D_{S1} \).

### 3.2.3.3 Terminal period

The policymaker enters period 2 with an outstanding debt \( D_{L0} + D_{S1} \), and the income \( y_2 \) is revealed. It is the last period and there is no additional borrowing; it is the only period when default is possible. If there is no default \( (\chi_2 = 1) \), the last-period consumption equals national income net of the repaid debt. In case of default \( (\chi_2 = 0) \), nothing is paid to the lenders, and the economy loses a fraction \( \mu \) of its income. The value function is therefore
the following:

$$V_2(y_2, D_{L0}, D_{S1}) = \max_{\chi_2 \in \{0, 1\}} u\left(\chi_2 \times (y_2 - D_{L0} - D_{S1}) + (1 - \chi_2) \times y_2(1 - \mu)\right). \quad (3.14)$$

One possible justification for the loss of a fraction of output in case of default is that trade sanctions may follow the default: Rose (2005) documents declines in international trade after sovereign defaults. Arellano (2008) argues that a sovereign default disrupts the supply of credit from the private financial sector, an essential factor of domestic output. Finally, a loss in output due to default is a necessary addition to make the punishment for default severe enough for the equilibrium amount of debt to be reasonably large, as pointed out by Aguiar and Gopinath (2006).

### 3.2.4 Optimal Policy

I use backward induction to solve for the optimal policy. Since there is no closed-form solution, I combine analysis with numerical optimization.

#### 3.2.4.1 Period 2

In the last period, the policymaker will simply decide whether to default or not and consume the entire income net of costs associated with debt. Equation (3.14) tells that the government will default if the cost of debt repayment exceeds the cost of default: $\chi_2 = 0$ if $D_{L0} + D_{S1} > \mu y_2$. Given that $y_H > y_L$, it is impossible to have a total debt payable in period 2 such that it is optimal to repay in case of low income and renege on debt in case of high income.

#### 3.2.4.2 Period 1

Because of the interaction between the chosen amount of short-term debt $D_{S1}$ and its price $q_{S1}$, the value function $V_1(\cdot)$ defined by equation (3.12) is not so straightforward to find. To assess the properties of $V_1(y_1, D_{L0}, D_{S0})$, let us restrict its state-space $S =$
\{y_L, y_H\} \times \mathbb{R}_+ \times \mathbb{R}_+ \text{ to a domain } S^R \text{ that can be considered as the result of optimal behavior in period 0. The value function } V_1(\cdot) \text{ can be defined, first, if the short-term bond price } q_{S_1} \text{ is consistent with the chance of default given the amount of debt to be repaid in the last period; and second, if consumption in period 1 is non-negative. Regarding the chance of terminal-period default and the corresponding bond price, equation (3.14) tells that there are three possibilities. First, if } D_{L_0} + D_{S_1} > \mu y_H \text{, the total debt payable in the last period is too high and the government is certain to default, so the bond price in period 1 is zero. Second, if } D_{L_0} + D_{S_1} \leq \mu y_L \text{, the debt is too small and the government will always repay in period 2, so the bond price is } \bar{q}_{S_1} \text{ (defined by equation (3.6)). Finally, if } \mu y_L < D_{L_0} + D_{S_1} \leq \mu y_H \text{, the government will prefer default only in case of low income in the last period, and the corresponding bond price is } \tilde{q}_{S_1} \text{ (equation (3.7)). Each of these three cases is possible for a different subset of state-space } S \text{ and must be considered separately.}

Let } S^A_{y_1} \text{ be the set of pairs } (D_{L_0}, D_{S_0}) \in \mathbb{R}_+^2 \text{ for which, given first-period income } y_1 \text{, there exists a choice of first-period short-term borrowing } D_{S_1} \in \mathbb{R}_+ \text{ such that first-period consumption is non-negative and the debt payable in the last period is so high that the government is guaranteed to default (so that the bond price } q_{S_1} \text{ is zero).}^9 \text{ Using (3.13) and (3.14), }

\begin{equation}
S^A_{y_1} = \{(D_{L_0}, D_{S_0}) \in \mathbb{R}_+^2 \mid y_1 - D_{S_0} \geq 0 \text{ and } D_{L_0} + D_{S_1} > \mu y_H, \text{ for some } D_{S_1} \in \mathbb{R}_+\},
\end{equation}

which can be simplified to become

\begin{equation}
S^A_{y_1} = \{(D_{L_0}, D_{S_0}) \in \mathbb{R}_+^2 \mid D_{S_0} \leq y_1\}.
\end{equation}

That is, if the sovereign can repay the debt } D_{S_0} \text{ out of national income, it can afford to issue so many new bonds that they guarantee default in the last period and their price is}

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9The superscripts stand for the likelihood of default in the last period: "Always", "Never", "Sometimes".
zero.

Similarly, let $S_{y_1}^N$ contain points $(D_{L0},D_{S0}) \in \mathbb{R}_+^2$ for which, given first-period income $y_1$, there is a value $D_{S1} \in \mathbb{R}_+$ small enough so that the government will never default in period 2, but at the same time large enough so that consumption is non-negative in period 1. Using (3.13) and (3.14),

$$S_{y_1}^N = \{(D_{L0},D_{S0}) \in \mathbb{R}_+^2 \mid y_1 - D_{S0} + \bar{q}_{S1}D_{S1} \geq 0 \text{ and } D_{L0} + D_{S1} \leq \mu y_L, \text{ for some } D_{S1} \in \mathbb{R}_+\},$$

which is simplified to become

$$S_{y_1}^N = \{(D_{L0},D_{S0}) \in \mathbb{R}_+^2 \mid D_{L0} \leq \mu y_L \text{ and } D_{S0} \leq y_1 + \bar{q}_{S1}(\mu y_L - D_{L0})\}. \quad (3.16)$$

Finally, let $S_{y_1}^S$ contain points $(D_{L0},D_{S0}) \in \mathbb{R}_+^2$ for which, given first-period income $y_1$, there is a value $D_{S1} \in \mathbb{R}_+$ that makes it optimal to default only in case of low income in period 2 and allows for non-negative consumption in period 1:

$$S_{y_1}^S = \{(D_{L0},D_{S0}) \in \mathbb{R}_+^2 \mid y_1 - D_{S0} + \bar{q}_{S1}D_{S1} \geq 0 \text{ and } \mu y_L < D_{L0} + D_{S1} \leq \mu y_H, \text{ for some } D_{S1} \in \mathbb{R}_+\},$$

which is simplified to become

$$S_{y_1}^S = \{(D_{L0},D_{S0}) \in \mathbb{R}_+^2 \mid D_{L0} \leq \mu y_H \text{ and } D_{S0} \leq y_1 + \bar{q}_{S1}(\mu y_H - D_{L0})\}. \quad (3.17)$$

The value function $V_1(\cdot)$ can be defined over the union of all subsets $S_{y_j}^i \times \{y_j\}$, where $i \in \{A,S,N\}$ and $j \in \{L,H\}$. To further simplify the problem, note that optimal policy will never lead to inadvertent default in the terminal period, because of the following two results. First, given $D_{L0} < \mu y_H$, the policymaker will never choose the amount of short-term debt in period 1 so high that default is guaranteed in period 2: $D_{L0} + D_{S1} \leq \mu y_H$. And
second, the policymaker will never choose the amount of long-term debt $D_{L0}$ in period 0 so high that default is guaranteed in period 2: $D_{L0} \leq \mu y_H$. To prove the first result, notice that setting the short-term debt in period 1 too high ($D_{S1} > \mu y_H - D_{L0}$) makes default unavoidable, so the lenders will assign zero price to such debt. Such debt does not increase consumption in period 1 but guarantees that the economy will incur the cost of default in period 2. There is always a better policy. For an example, given $D_{L0} < \mu y_H$, if the policymaker sets $D_{S1} = (\mu y_H - D_{L0})/2$, the outright gain in period 1 is $q_{S1} D_{S1}$ (where $q_{S1}$ is positive because the debt is likely to be repaid), and the cost of debt repayment $D_{S1} + D_{L0}$ in period 2 is less than the cost of default $\mu y_H$, at least in case of high-income state. A similar argument proves the second statement. The long-term debt will be less than $\mu y_H$: it can be sold at a positive price $q_{L0}$ and increase consumption in period 0; also, given the first result, it allows to increase consumption in period 1 by issuing short-term debt $D_{S1}$ at a positive price $q_{S1}$; and finally, the cost of debt repayment $D_{L0} + D_{S1}$ will be less than the cost of default in period 2, at least in case of high-income state.

For the purposes of finding the optimal policy, the two results allow to restrict the domain of the value function $V_1(\cdot)$ to $S^R = \{S^i \times y_j \ | \ i \in \{S, N\}, j \in \{L, H\}\}$. Over this domain, the value function can be found as follows. Let $V_1^S(\cdot)$ be the first-period value function consistent with default that happens in case of low-income state in period 2:

$$V_1^S(y_1, D_{L0}, D_{S0}) = \max_{D_{S1}} u(c_1) + \beta E[V_2(y_2, D_{L0}, D_{S1})]$$

s.t.

$$c_1 \leq y_1 - D_{S0} + \tilde{q}_{S1} D_{S1};$$

$$\mu y_L < D_{L0} + D_{S1} \leq \mu y_H.$$

Similarly, the value function $V_1^N(\cdot)$ is consistent with the choice of $D_{S1}$ small enough so that the government will never default:

$$V_1^N(y_1, D_{L0}, D_{S0}) = \max_{D_{S1}} u(c_1) + \beta E[V_2(y_2, D_{L0}, D_{S1})]$$
s.t. \( c_1 \leq y_1 - D_{S0} + \bar{q}S_1D_{S1}; \)
\[ D_{S1} + D_{L0} \leq \mu y_L. \]

Given these definitions, the value function (3.12) can be expressed as

\[
V_1(y_1, D_{L0}, D_{S0}) = \begin{cases} 
V^N_1(y_1, D_{L0}, D_{S0}), & \text{if } (D_{L0}, D_{S0}) \in S^N_{y_1} \setminus S^S_{y_1}, \\
V^S_1(y_1, D_{L0}, D_{S0}), & \text{if } (D_{L0}, D_{S0}) \in S^S_{y_1} \setminus S^N_{y_1}; \\
\max \{V^N_1(y_1, D_{L0}, D_{S0}), V^S_1(y_1, D_{L0}D_{S0})\}, & \text{if } (D_{L0}, D_{S0}) \in S^N_{y_1} \cap S^S_{y_1}. 
\end{cases}
\] (3.18)

The provided formulation tells how to numerically estimate the value function \( V_1(\cdot) \).

Figure 3.1 schematically summarizes the results of numerical evaluation for a given income \( y_1 \) (the results are essentially the same for low and high income). Area ODEF represents the subset \( S^N_{y_1} \) compatible with zero chance of default in the last period; area OABC corresponds to the subset \( S^S_{y_1} \) compatible with likely default. Notice that defaultable debt, when it becomes available, expands the set of choices for the government towards greater debt: in the figure, area OABC expands the domain of the value function. Shaded area represents the states where large debt \( D_{L0} + D_{S1} \) and likely default is the optimal policy; white area stands for the points where smaller debt with zero chance of default is preferred. Notice that the latter is bounded by a downward-sloping line DGF: quite intuitively, a large current debt may trigger an optimal policy that makes default likely in the future. Finally, note that the value functions \( V^N_1(\cdot) \) and \( V^S_1(\cdot) \) are continuous and differentiable over their respective domains \( S^N_{y_1} \) and \( S^S_{y_1} \). Because \( V_1(\cdot) \) is a maximum of the two values supplied by \( V^N_1(\cdot) \) and \( V^S_1(\cdot) \), it is continuous but not differentiable (along the line GF in figure 3.1), which complicates the numerical optimization.

Figure 3.2 demonstrates the typical shape of the first-period value function \( V_1(\cdot) \) parameterized and plotted for a given \( y_1 \). For illustrative purposes, I have evaluated the function over all the points where it can be defined; notice however, that the restricted
domain $S_{y_1}^R$ only includes the points that can correspond to a likely debt repayment in the last period ($D_{L0} \leq \mu y_H$).\footnote{The value function $V_1(\cdot)$ can be defined and is finite for any level of $D_{L0} \geq 0$ as long as $D_{S0} < y_1$. Any such state is a part of subset $S_{y_1}^A$ that corresponds to the points compatible with a policy that makes default in period 2 certain. I have excluded this subset from the domain of the value function as sub-optimal.} Naturally, the value function is non-increasing in both short-term and long-term debt, since the sovereign incurs nothing but costs of debt after the period when the debt is issued. It is clear from the figure that the value function is more sensitive to short-term than long-term debt. By construction, short-term debt $D_{S0}$ cannot be repudiated; it is a non-contingent payment that directly affects consumption in period 1. High short-term debt $D_{S0}$ eventually drives the first-period consumption to zero and the value function $V_1(\cdot)$ to negative infinity. As for long-term debt, it can be repudiated in the last period, so its impact on consumption path and on the value function is limited.

What prevents the policymaker from borrowing an infinite long-term debt and enjoying
infinite consumption in period 0 is the fact that lenders would price such irrecoverable debt accordingly. Therefore, in period 0, whether the policymaker finds it optimal to borrow long-term so extensively that default becomes likely depends on the supply of debt.

Figure 3.2: Parameterized and numerically evaluated value function $V_1(y_1, D_{L0}, D_{S0})$. Lighter shade stands for higher value, white area stands for points where it is undefined. The restricted domain $S^R_{y_1}$ is the set of points for which the value function is defined, with the exception of points that are only compatible with certain default; $S^R_{y_1}$ is represented by the shaded area to the left from the dashed line.

### 3.2.4.3 Period 0

The policymaker must decide upon the optimal debt combination $\{D^*_{L0}, D^*_{S0}\}$ that maximizes the value function $V_0(\cdot)$ defined by equation (3.10). The choices $\{D_{L0}, D_{S0}\}$ that can be considered as candidates for optimal policy must be elements of $(S_N^{y_L} \cup S_S^{y_L}) \cap (S_N^{y_H} \cup S_S^{y_H})$, where sets $S^N_{y_1}$ and $S^S_{y_1}$ are defined by equations (3.16) and (3.17). In other words, the policy chosen in the initial period must be such that it falls within the restricted state domain of the value function $V_1(\cdot)$ for any first-period income $y_1$, high or low. Let us define a combination $\{D_{L0}, D_{S0}\}$ as feasible if it satisfies this condition, as well as the budget constraint (3.11). Like in the first period, there is interaction between bond prices. On the
one hand, given that a policy is feasible, it does not have impact on the short-term bond price $q_{S0}$, because the short-term debt that matures in period 1 is assumed to be repaid and the likelihood of high or low income in period 1 is exogenous. On the other hand, a policy $\{D_{L0}, D_{S0}\}$ chosen in period 0 defines whether default in period 2 is likely after the economy experiences high or low income in period 1; the bond price $q_{L0}$ must reflect this fact. This interaction complicates the search for the global maximum of the value function $V_0(\cdot)$.

However, I find it sufficient for illustrative purposes to evaluate the value function $V_0(\cdot)$ for all the feasible values of long-term debt $D_{L0}$ and their corresponding feasible choices of short-term debt $D_{S0}^*|D_{L0}$ that maximize the value function $V_1(\cdot)$ given fixed $D_{L0}$. As equations (3.11), (3.16), and (3.17) suggest, it is possible to find a feasible pair $\{D_{L0}, D_{S0}\}$ only for $D_{L0} \in [0, \mu_y H]$, as long as the initial debt $D_0$ is not too large. For a given value $D_{L0}$, the computational strategy to find $V_0(\cdot)$ is the following:

1. Find the price of the short-term debt, $q_{S0}$.

2. Given $D_{L0}$, use numerically estimated $V_1(\cdot)$ and its definition (3.18) to find the ranges of $D_{S0}$ that are compatible with the following four cases of likelihood of default:

   (a) default is never likely:

   $$V_1(y_L, D_{L0}, D_{S0}) = V_1^N(y_L, D_{L0}, D_{S0}), \quad V_1(y_H, D_{L0}, D_{S0}) = V_1^N(y_H, D_{L0}, D_{S0});$$

   (b) default is likely only after the first period with low income:

   $$V_1(y_L, D_{L0}, D_{S0}) = V_1^S(y_L, D_{L0}, D_{S0}), \quad V_1(y_H, D_{L0}, D_{S0}) = V_1^N(y_H, D_{L0}, D_{S0});$$

   (c) default is likely only after the first period with high income:

   $$V_1(y_L, D_{L0}, D_{S0}) = V_1^N(y_L, D_{L0}, D_{S0}), \quad V_1(y_H, D_{L0}, D_{S0}) = V_1^S(y_H, D_{L0}, D_{S0});$$
(d) default is likely regardless of the first-period income:

\[ V_1(y_L, D_{L0}, D_{S0}) = V_1^S(y_L, D_{L0}, D_{S0}), \quad V_1(y_H, D_{L0}, D_{S0}) = V_1^S(y_H, D_{L0}, D_{S0}). \]

3. For all four cases, find the corresponding price \( q_{L0} \) using equations (3.6)–(3.8).

4. Find the value \( D_{S0} \) that maximizes the value function \( V_0(\cdot) \) in each of the four ranges.

5. Compare the four values of the value function to find the maximum; choose \( D_{S0}^* | D_{L0} \), the optimal short-term debt given \( D_{L0} \).

The following section delivers the results of the computational exercise.

### 3.3 Results

#### 3.3.1 Parameter Specification

Table 3.1 summarizes the parameter values selected to represent the baseline case. The time-preference parameter is set to be \( \beta = 0.95 \). I assume that the lenders have the same patience as the policymaker, so that the risk-free rate is defined as \( r = 1/\beta - 1 \). I set low income to be equal to only a third of high income: \( y_H = 1; y_L = 1/3 \). Such income variability is quite dramatic. In an infinite-horizon dynamic stochastic model, one would usually have to associate the default with an extremely large negative persistent income shock (or a significant decline in trend), or a sequence of negative shocks. The whole point of borrowing is to smooth consumption; if there is little income volatility, the sovereign would never borrow so much that the default is likely, because it sharply diminishes the bond prices and does not help smooth the path of consumption. The need to have a possibility of default combined with the lack of horizon in a model with three periods justifies my choice of income variability. In addition to creating the possibility of default, such volatile income process helps study the impact of lenders’ and policymaker’s risk-aversion. High risk-aversion puts more value to smooth consumption path on the policymaker’s side and
Table 3.1: Parameter values of the baseline model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.95</td>
<td>Policymaker’s discount factor</td>
</tr>
<tr>
<td>$y_H$</td>
<td>1.0</td>
<td>High income</td>
</tr>
<tr>
<td>$y_L$</td>
<td>0.33</td>
<td>Low income</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.33</td>
<td>Cost of default</td>
</tr>
<tr>
<td>$\eta$</td>
<td>1.0</td>
<td>Risk-aversion coefficient of the policymaker</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.5</td>
<td>Sensitivity of lenders’ stochastic discount factor to income</td>
</tr>
<tr>
<td>$s$</td>
<td>0.04</td>
<td>Persistence of the persistent income process</td>
</tr>
<tr>
<td>$p$</td>
<td>0.04</td>
<td>Prior likelihood of the persistent income process</td>
</tr>
<tr>
<td>$D_0$</td>
<td>0.5</td>
<td>Initial debt</td>
</tr>
</tbody>
</table>

The cost of default, measured as a fraction of default-state income, is set to be $\mu = 1/3$. This cost is set high enough so that there is significant amount of debt maturing in period 2 that can be credible. A low value of parameter $\mu$ would make a commitment to repay any significant amount of long-term debt non-credible, whereas the most interesting characteristic of the equilibrium is the maturity of debt issued in the initial period.

All the remaining parameter values are specific only to the baseline case; their impact on the equilibrium is subject to scrutiny. The policymaker has a CRRA utility function $u(c) = c^{1-\eta}/(1-\eta)$ and the risk-aversion coefficient $\eta$ is equal to one, so that the utility function is logarithmic in the baseline case. Parameter $\alpha$ defines the dependence of lenders’ stochastic discount factor on the economy’s income. When $\alpha = 0$, the lenders are neutral to the risk specific to the borrowing economy: $E_t[m_{t+1} \mid y_t] = 1/(1+r)$. In this case, bond prices simply correspond to the risk-free rate adjusted for chance of default. A higher

\[^{11}\text{At the same time, low income cannot be equal to zero, because it would create a chance of zero consumption in the terminal period, which would make the specified utility function become infinite.}\]
value of $\alpha$ corresponds to greater risk-aversion and lower bond prices. I set $\alpha = 1/2$ as the baseline case. In the initial period, agents suspect that the income process may be represented by a persistent Markov chain, and have a prior belief that this scenario is the case with probability $p$. As a starting point, let $p = 0.04$; and let the parameter that defines the persistence of the persistent Markov chain be $s = 0.04$, which, as described in section 3.2.1, yields the following competing descriptions of the income process:

$$P_1 = \begin{pmatrix} 0.98 & 0.02 \\ 0.02 & 0.98 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix},$$

where transition matrix $P_1$ represents the persistent-income scenario. Finally, the debt payable immediately in period 0 is set to be $D_0 = 1/2$, which exceeds the income in the initial period. By means of such high debt, the economy is put into a dire situation where it cannot afford to repay the debt using its income and has to borrow extensively.

### 3.3.2 Results: Baseline Case

Figure 3.3 illustrates the optimal policy in the baseline case. It shows the optimal short-term debt in the initial period for a given amount of long-term debt $D^*_{S0}|D_{L0}$, as well as the corresponding value function. There are three lessons to be learned from these graphs.

First, the baseline-case optimal policy is for the policymaker to have zero long-term debt. In equilibrium, the sovereign uses only the short-term debt to create consumption in period 0 and then refineses depending on the state of income in period 1.

Second, it is clear that, for a larger fixed amount of long-term debt, the optimal policy shifts towards a likelier default. As $D_{L0}$ increases, non-zero chance of default becomes either a preferred choice or the only option available. For example, consider the switch from absolutely safe debt to the debt that will be defaulted if $y_0 = y_1 = y_2 = y_L$. As the exogenous value of the long-term debt approaches the threshold between the two types of
Figure 3.3: Initial-period value function and optimal short-term debt for a given long-term debt $D_{L0} \in [0, \mu y_H]$. Lines marked with circles represent choices that never lead to default. Asterisks mark choices that may lead to default if income remains low in all three periods. Diamonds mark choices that will result in default whenever there is low income in the last period, regardless of the state of income in the middle period.

Cutting short-term debt in period 0 is a way to commit against additional borrowing in period 1, which keeps the debt that matures in period 2 safe. The cut causes the consumption in period 0 and the value function $V_0(\cdot)\mid D_{L0}$ to fall dramatically, until the low-priced but extensive risky debt becomes preferable. As for the transition from the debt that is defaulted only when $y_0 = y_1 = y_2 = y_L$ to the policy that leads to default whenever $y_2 = y_L$, it happens because the latter becomes the only option for $D_{L0} > \mu y_L$.

Finally, for a given likelihood of default, the relationship between a given long-term debt $D_{L0}$ and the corresponding optimal short-term debt $D^{*}_{S0} \mid D_{L0}$ is negative: when it comes to financing immediate consumption, the bonds of different maturities are substitutes. A

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$^{12}$Figure 3.1 demonstrates why it is the case: the border between the safe and risky debt, indicated by curve $GF$, is downward-sloping; when it is steep, in order to remain within the safe-debt region, the policymaker eventually has to cut the short-term debt sharply when the long-term debt rises.

$^{13}$Chatterjee and Eyigungor (2012) argue that short-term debt is generally cheaper to the borrower (it has a lower yield, or higher price) precisely because the long-term debt suffers from lack of commitment: there is always an incentive for additional borrowing before the long-term debt matures, which leads to higher default risk.
more interesting feature is the slope of the value function for a given likelihood of default. Consider the region of default-free debt first: when the long-term debt is safe, its price is high and it creates large gains to consumption in period 0. However, the value function \( V_0(\cdot)|D_{L0} \) is the highest for small amounts of safe long-term debt. The main reason is that safe long-term bonds require that the borrower makes non-contingent payments in the terminal period. What changes when the long-term debt is not safe? First, the long-term bond price \( q_{L0} \) is lower due to a positive chance of default; therefore, long-term bond adds little to consumption in period 0 and to the value function \( V_0(\cdot)|D_{L0} \). However, as figure 3.3 clearly indicates, the policymaker can actually benefit from increasing the scope of long-term borrowing in case when the long-term debt is risky (at least in case when default occurs if \( y_0 = y_1 = y_2 = y_L \)). The reason is that, while more long-term debt is associated with greater consumption in period 0, its impact on the cost of debt is limited because the government does not repay the debt in case of low income in the last period. For the case when default occurs whenever the last-period income is low \( (y_2 = y_L) \), the price of long-term debt is too low for the government to benefit from its expansion. Actually, this level of default risk is never an equilibrium outcome; this result holds for all calibrations studied below.

It is necessary to make a remark regarding the simplifying assumption that the government cannot save \( (D_i \geq 0) \). This assumption puts long-term borrowing in conflict with consumption-smoothing and therefore makes it less attractive. Long-term debt decreases consumption in the last period, when the long-term debt is to be repaid. In the intermediate period, it shifts the optimal policy towards saving, which is not an option by assumption. In this respect, having more short-term debt \( D_{S0} \) may be better because it is repaid in period 1, and the policymaker can rely on additional borrowing \( D_{S1} \) to smooth consumption. This is especially important in the case when the debt is safe, and less so when the government defaults in low-income state. Effectively, this assumption shifts the optimal policy towards short-term borrowing in the initial period. However, the main purpose of the model is to determine the factors that are important for the choice of debt
maturity, and the no-saving assumption does not qualitatively affect the analysis provided below. The benefit of this assumption is that it simplifies the solution.

### 3.3.3 Lessons from the Model

#### 3.3.3.1 Does Learning Matter?

The central question is, of course, whether uncertainty about the persistence of income process and Bayesian learning matter for the optimal debt maturity in equilibrium. To answer this question, I solve the model for the baseline parameter specification but in an alternative setting where Bayesian updating does not happen. Instead, the initial prior describes the estimated probabilities of future income states both in period 0 and period 1: equations (3.2) and (3.3) are replaced with \( l_L = l_0 \) and \( l_H = 1 - l_0 \). Figure 3.4 demonstrates the results. Clearly, Bayesian updating tilts the policymaker’s preference towards shorter maturity and safer bonds. In the top-left panel, the value function \( V_0(\cdot)|D_{L_0} \) corresponding to the baseline case with Bayesian learning is higher for the range of safe long-term debt \( D_{L_0} \), and lower for the range corresponding to likely default. In fact, numerical estimation shows that in the baseline case, the optimal policy in equilibrium is zero long-term debt and no chance of default, while in absence of Bayesian learning, there is substantial long-term debt that is likely to be repudiated.

The top-right panel of figure 3.4 shows how Bayesian learning affects the updated prior in period 1. Without learning, the state of income in the intermediate period does not change the agents' belief about the nature of the income process; agents stick to the prior and believe that income in the second period will be low with probability \( l_0 \) if income in period 1 is low and with probability \( 1 - l_0 \) if income in period 1 is high. With learning, the state of income in the first period reveals additional information about the income process. Namely, if income in period 1 is low, income process is likelier to be persistent and remain low, so \( l_L > l_0 \). If income in period 1 is high, the income process is likelier to be transitory and switch back to low income in period 2, so \( l_H > 1 - l_0 \). Of course, if the
Figure 3.4: Bayesian learning tilts the optimal policy towards less long-term debt and a less likely default. Bayesian learning \([1]\) decreases the value function for policies associated with likely default and increases it for policies corresponding to default-free debt (top left); \([2]\) makes agents update their beliefs in period 1 so that low income in period 2 is likelier (top right); \([3]\) makes prices for short-term safe bonds \(\bar{q}_{S1}\) higher and price for risky bonds \(\tilde{q}_{S1}\) lower in period 1 (bottom left); and \([4]\) increases the wedge between the price for long-term safe bonds \(\bar{q}_{L0}\) and the price for long-term risky bonds \(\tilde{q}_{L0}\) in period 0 (bottom right).

agents are initially absolutely certain that the income process is persistent or transitory (i.e., \(p = 1\) or \(p = 0\)), the prior is not updated in period 1, which is equivalent to no learning. In short, Bayesian learning increases the estimated likelihood of the low-income state in period 2 regardless of the state of income in period 1. This, in turn, makes short-term debt preferable in period 0, by means of bond prices assigned by lenders and
consumption-smoothing executed by the borrower.

The two bottom panels of figure 3.4 indicate how learning affects the bond prices. By assumption, investors are risk-averse; they put greater value on payments they receive in low-income states. Because Bayesian updating increases the likelihood of low income in the last period, it increases the price $\bar{q}_{S1}$ of safe short-term bonds that are issued in period 1 and are not likely to lead to default. For the same reason, it decreases the price $\bar{q}_{S1}$ of risky short-term debt that is only repaid in case of high income in the last period. Also, Bayesian learning affects the long-term bond prices in period 0, which are simply discounted expected short-term bond prices from period 1. As the bottom-right panel of the figure shows, learning increases the wedge between the prices for safe and risky long-term bonds. To sum up, Bayesian learning makes safe bonds relatively more expensive (which corresponds to cheaper borrowing) than risky bonds.

As for the sovereign, commitment to repayment is possible if most of the deficit in the initial period is financed by means of short-term bonds that mature in the intermediate period. Consequently, in the intermediate period, the consumption-smoothing motive will require that the policymaker refinances by issuing new short-term bonds. In presence of learning, the policymaker can commit to debt repayment by keeping the total debt maturing in the last period small enough and refinance more effectively in the intermediate period by selling short-term safe bonds that will be relatively more expensive. On the other hand, learning makes it problematic for the sovereign to use an extensive risky long-term debt to finance consumption. First, risky long-term bonds are relatively cheaper due to learning. And second, additional borrowing in the intermediate period, which is much needed for a smooth consumption path in case of low income, becomes painful due to lower price of new short-term risky bonds.

Intuitively speaking, when nobody is certain about the persistence of an unfolding recession, two things happen. On the investors’ side, there is little desire to grant a large long-term debt. On the upside, the recession is short and the economy will quickly recover and repay the debt eventually. In the worst-case scenario, the recession is long and may
eventually lead to a default. It is only with time that lenders can learn which scenario is the case, so that uncertainty adds price risk to long-term bonds when default is a possibility. Price risk is not a concern for short-term bonds, which mature early, before anyone believes the economy may default. On the borrower’s side, additional price risk inherent to long-term bonds simply makes it cheaper to shift towards short-term borrowing and subsequent rollovers. As long as default is not an immediate concern, short-term borrowing will remain cheaper even if the recession turns out to be persistent. The reason is that the lenders’ portfolio returns are correlated with the economy’s performance, so that lenders value payments they receive in bad states of the economy. In line with the literature, the reason for the maturity of sovereign debt to shorten during recessions is the supply-side shift that makes short-term debt more accessible.\textsuperscript{14}

Note that bond prices in the model can exceed unity for high values of $p$. This occurs because investors do not have any alternative risk-free saving technology, such as cash under the pillow. However, for the purposes of the qualitative analysis, it is sufficient to restrict the range of $p$ to a set of small values (up to 0.05), for which it is not a problem. Moreover, despite the fact that the parameters of this simple model are not directly related to an infinite-horizon dynamic stochastic case, it is worth noting that empirical estimations (such as Aguiar and Gopinath (2004)) suggest that the persistent component of a typical income process that describes an emerging economy’s GDP has a very small variance compared to that of the transitory component. This means that any income shock is interpreted upon its impact as very unlikely to be persistent.

Also, note that Bayesian learning may seem to have little impact on prices of bonds, especially the long-term ones. Still, the effect on bond prices, combined with the prior updated in favor of low income in the last period, is sufficient to have a definitive impact on the optimal debt maturity in equilibrium. In addition, the degree to which learning affects the bond prices depends on lenders’ risk-aversion, and its variations significantly affect the equilibrium, as shown below.

\textsuperscript{14}For example, see Broner et al. (2013).
3.3.3.2 Sharing the Risk

Risk-aversion, both on the lenders’ and on the policymaker’s side, is important for the optimal debt maturity in equilibrium. Parameter $\alpha$ measures the degree to which the lenders’ stochastic discount factor is correlated with the economy’s income. According to equation (3.4), if $\alpha = 0$, lenders are not sensitive to the state of income of the borrowing economy. A higher $\alpha$ makes lenders assign greater value to payoffs they receive in cases when the borrowing economy exhibits low income. If the government debt is likely to be repudiated, the payoffs from bonds are correlated with income. It is therefore safe to say that parameter $\alpha$ measures the lenders’ aversion to risk associated with payoffs from holding government bonds. So, risk-averse investors will dislike extensive long-term lending that may lead to default; risky bonds will be priced accordingly, and in equilibrium, the policymaker will find it preferable to rely more on short-term bonds. It is prices that shift the optimal policy towards a sequence of short-term borrowing and a smaller chance of default in the terminal period. This effect is reinforced by Bayesian learning, which increases the updated likelihood of the low income in the last period: a likelier low-income state makes default risk costly to the borrower, while safe bonds are relatively attractive.

Left panel of figure 3.5 demonstrates the effect of lenders’ risk-aversion on the outcome of the model. Higher values of $\alpha$ increase the value function that corresponds to policies involving little long-term debt and no chance of default in the terminal period. At the same time, it decreases the prices of risky bonds, which causes the value function associated with extensive long-term borrowing and likely default to drop. The result is that higher risk-aversion on the lenders’ side moves the equilibrium in the initial period towards shorter debt maturity.

Parameter $\eta$ stands for the risk-aversion coefficient of the policymaker’s CRRA utility function. It does not affect the bond prices, so it affects the optimal policy in the initial period only to the degree to which different debt combinations differ in their ability to smooth consumption. As the right panel of figure 3.5 reveals, a higher $\eta$ makes the value
Figure 3.5: Effect of lenders’ and policymaker’s risk-aversion on the optimal policy in the equilibrium. More risk-averse investors will evade long-term bonds that are subject to price risk, so that the equilibrium will shift towards shorter maturity (left panel). More risk-averse policymaker will prefer long-term debt that can be repudiated (right panel).

function lower for policies involving absolutely no chance of default; at the same time, given a non-zero chance of default, it makes the policymaker benefit more from expanding the long-term borrowing. A more risk-averse government will be more eager to rely on extensive long-term debt accompanied by a likely default in the terminal period. The reason is, again, that the impact of a risky debt on the last-period consumption is limited; there is always an option to renege on debt. Default protects the risk-averse policymaker against cases of extremely low consumption.

These results are in line with Broner et al. (2013) who argue that the optimal maturity is a result of a risk-sharing equilibrium: risk-averse lenders like short-term bonds because they protect against credit and price risk; risk-averse government likes long maturity because it allows for better consumption-smoothing. I also find that in presence of default risk, the sovereign prefers to borrow long-term and default in low-income state in the last period. Short-term debt has to be refinanced in the intermediate period, which is very costly if the income in the middle period is low and it starts to look like the recession is persistent.
and default is a possibility. What is different in my model is that default is a strategic decision, which is not necessarily likely in equilibrium.\footnote{Broner et al. (2013) focus on risky debt only; their model guarantees default in a low-income state in the last period.} If lenders are more risk-averse, they charge a large risk premium on defaultable long-term bonds and price the sovereign out of extensive risky long-term borrowing. In this respect, the results are close to the argument by Chatterjee and Eyigungor (2012) that risk-averse investors prefer short-term bonds because they can be used more efficiently to enforce the sovereign’s commitment to repay the debt. In addition, I stress that uncertainty about the future of the economy in a recession can add to risk premiums and shorten the equilibrium maturity of sovereign debt.

### 3.3.3.3 Income Process

The initial period of the model represents an unfolding recession of an uncertain length, with agents considering worst-case and best-case scenarios about the future of the economy. The common knowledge is that if the economy borrows extensively and then suffers a long recession, default will become a possibility. Intuitively, then, if the worst-case scenario is a particularly long recession, and if this scenario is considered likely, the outcome is that the government must have hard times procuring a large long-term debt. Figure 3.6 shows that the model is in line with this intuition.

First, the optimal policy in equilibrium shifts towards short-term debt when the prior belief is that a persistent income process is likely. The reason is that, for higher $p$, the low-income state in the terminal period is likelier, so that a long-term debt so extensive that it may be repudiated becomes very expensive to the borrower due to a lower price $\bar{q}_{L0}$. At the same time, the price of safe long-term bonds $\bar{q}_{L0}$ becomes higher, since these bonds pay off in a low-income state that is likelier. Note that the effect on the equilibrium is dramatic even for small values of $p$. This suggests that, in an infinite-horizon dynamic stochastic model, the presence of a trend component in the aggregate income process has
Figure 3.6: Effect of persistence $s$ and likelihood $p$ of the persistent income process on the optimal policy in the equilibrium. A likelier persistent income process makes it optimal to borrow short-term (left panel). A more persistent income process makes it optimal to borrow short term; $s = 1$ stands for the two competing descriptions of income process being the same: $P_1(i, j) = P_2(i, j) = 1/2$ (right panel).

These mechanics imply that uncertainty and learning have a potential to distinguish a large impact on the optimal debt maturity, even though its variance may be relatively small (which seems to be the case even for emerging economies, as documented by Aguiar and Gopinath (2004)).

Second, the equilibrium debt maturity is shorter when the worst-case scenario is a very persistent income process. Setting $s$ equal to 1 means that the persistent Markov chain is described by the same transition matrix as the transitory one ($P_1 = P_2$), so that the income process is certainly transitory. For $s = 0$, the persistent income process is described by a unit transition matrix: $P_1 = I$. If the transition matrix $P_1$ has eigenvalues close to one, the implication is that, if uncertainty about the nature of the income process resolves in favor of the persistent process in period 1, the low-income state in the last period is highly probable, which ultimately increases the chance of default in the last period for an extensive long-term debt and adds price risk to long-term bonds.
between the choices of debt maturity optimal for policymakers governing emerging and high-income economies in distress. An emerging economy is more likely to experience persistent shocks to aggregate income. Therefore, when a recession unfolds in such economy, investors worry about a possible trend reversal and price long-term bonds accordingly. The outcome is that emerging economies exhibit strongly pro-cyclical government debt maturity, unlike high-income economies. The traditional assumption in the literature is that developed economies do not default, implying that these economies should exhibit less pro-cyclical variation in debt maturity. Historically, sovereign defaults have been limited to emerging economies. However, the recent events have shown how small is the circle of ‘safe-heaven’, default-free developed countries. During the Great Recession, many countries faced a sharp drop in sovereign credit ratings—Ireland, Spain, Portugal, to name a few. Greece experienced a sovereign debt crisis. Many of the troubled countries were widely considered as developed economies. At least in the investors’ eyes, default in a developed economy seems to be possible. The introduced model can explain pro-cyclical variation of debt maturity in an emerging economy and the lack of such in a developed economy by differences between the typical income processes that describe these economies and that are empirically documented in the literature.

3.3.3.4 Initial Debt

Figure 3.7 demonstrates the importance of the initial debt $D_0$ with which the economy is born. For a higher initial debt $D_0$, the (normalized) value function $V_0(\cdot)|D_{L,0}$ becomes more sensitive to the level of long-term debt $D_{L,0}$ over the region where long-term debt is likely to be repudiated. Eventually, for $D_0$ high enough, issuing a significant amount of long-term debt subject to default risk becomes preferred over having only short-term debt in the initial period and no chance of default in the last period. Notice that the amount of debt $D_0$ payable in the initial period for which extensive long-term borrowing and a likely default become optimal is well in excess of that period’s income $y_L$.

Essentially, higher initial debt reduces the funds available for consumption and effec-
tively makes the borrower more risk-averse. A more risk-averse borrower prefers extensive long-term debt subject to default risk, as explained above. If the burden of debt is heavy, paying the debt is particularly costly when income is low, so the sovereign accepts a possibility of default despite the risk premium it creates.

3.4 Final Remarks

3.4.1 Future Extensions

The model studies a scenario of a recession of uncertain persistence and establishes that such uncertainty is capable of shortening the debt maturity. It is more so for an economy where a persistent income process is particularly persistent or likely. These conclusions are in line with the evidence from emerging economies, which are documented to exhibit short debt maturity during recessions (Arellano and Ramanarayanan, 2012; Broner et al., 2013) and to have income processes with significant trend components, contrary to developed economies (Aguiar and Gopinath, 2004). As discussed in section 3.3.3.3, these results also imply that a developed economy, if it is subject to default risk, should not experience
such dramatic cuts in maturity during recessions, because its income process has a smaller persistent component.

Hence, an interesting question is whether uncertainty and learning about the persistence of income shocks can account for differences in dynamics of debt maturities of the two types of economies once the assumption about default-free developed economies is relaxed. In addition, it is interesting to investigate into the ability of learning to improve the empirical performance of a model of a small open economy with endogenous maturity of debt.

Therefore, an infinite-horizon dynamic stochastic model that can be tested against the data seems to be an interesting extension. However, endogenous treatment of debt maturity requires a high-dimensional state vector, which complicates the solution of the model. In addition, the problem is exacerbated by the fact that income process should have two components of different persistence for uncertainty about the persistence of income shocks to exist. The maturity structure of debt can be simplified down to two state variables in order to account for it in a tractable manner; Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012) design such simplifications. To keep the number of continuous state variables low, the income state can be discretized using the algorithm by Tauchen and Hussey (1991). The discretization is complicated due to the fact that income process must be driven by two shock processes of different persistence and due to uncertainty about the source of income fluctuations. A recent work by Judd et al. (2011) shows a promising direction to take in order to solve a model with strategic default in a high-dimensional set-up.

3.4.2 Conclusion

In line with the existing literature on sovereign debt maturity in emerging economies, the model delivers intuitive results regarding the importance of risk-aversion on both sides of the debt contract, the lack of credible commitment to repay inherent to long-term debt, and the importance of the size of the outstanding debt. In addition, it accounts for the fact

\footnote{See Chatterjee and Eyigungor (2012) for a discussion of the problem.}
that the sources of income dynamics may be of different persistence, introduces uncertainty about such persistence, and delivers the following two key results. First, uncertainty about the length of an unfolding recession embeds long-term bonds with additional risk and effectively shortens the equilibrium maturity of the sovereign debt. And second, uncertainty is more capable to shorten the debt maturity if it exists in presence of potentially very persistent income shocks and if such shocks are likely.

Therefore, without the need to assume that there is no default risk in developed economies, the provided theoretical framework can rely on typical features of income processes in emerging and developed economies in order to create much more pro-cyclical debt maturity in emerging economies. This idea requires further empirical investigation, though.
Appendix A

SUPPLEMENT TO CHAPTER 1

A.1 Model Solution

A.1.1 A Few Facts About the Default Threshold $\bar{\omega}$

For convenience, I provide several expressions related to $G(\bar{\omega})$ and $\Gamma(\bar{\omega})$ that are heavily used to solve the model. Let $\Phi(\cdot), \phi(\cdot)$, and $h(\cdot)$ be c.d.f., p.d.f., and hazard function of the standard normal distribution. The variable $\omega$ is log-normally distributed: $\ln \omega \sim N(-\sigma^2/2, \sigma^2)$. Define $z = \ln \bar{\omega}/\sigma + \sigma/2$; it is straightforward to show that $\phi(z - \sigma) = \bar{\omega}\phi(z)$ and $\partial \phi(z) / \partial z = -z\phi(z)$. Applying these facts to equations (1.2) and (1.3), I get

\[
\begin{align*}
\bullet G &= \Phi(z - \sigma) \Rightarrow \frac{\partial G}{\partial \bar{\omega}} = G' = \frac{\phi(z)}{\sigma} \Rightarrow \frac{\partial G'}{\partial \bar{\omega}} = -z\phi(z) / \bar{\omega} \sigma^2; \\
\bullet \Gamma &= \Phi(z - \sigma) + \bar{\omega}(1 - \Phi(z)) \Rightarrow \frac{\partial \Gamma}{\partial \bar{\omega}} = \Gamma' = 1 - \Phi(z) \Rightarrow \frac{\partial \Gamma'}{\partial \bar{\omega}} = -\phi(z) / \bar{\omega} \sigma
\end{align*}
\]

It is straightforward to show that $\Gamma'(\bar{\omega}_t) - \mu G'(\bar{\omega}_t) > 0$ around the steady state by combining the steady-state versions of equations A.6 and A.10.

A.1.2 De-trending

Let $A_{c,t}$ denote the following measure of total productivity in consumption good sector:

\[
A_{c,t} = A_{y,t}^{1-\alpha_y} A_{k,t}^{\alpha_y}.
\]  

(A.1)

Then, consumption and most other variables are de-trended using $A_{c,t}$:

\[
\begin{align*}
c_t &= \frac{C_t}{A_{c,t}}; \hat{c}_t = \frac{\hat{C}_t}{A_{c,t}}; b_t = \frac{B_t}{A_{c,t}}; \hat{b}_t = \frac{\hat{B}_t}{A_{c,t}}; w_t = \frac{W_t}{A_{c,t}}; \hat{w}_t = \frac{\hat{W}_t}{A_{c,t}}; x_t = \frac{X_t}{A_{c,t}}; \hat{x}_t = \frac{\hat{X}_t}{A_{c,t}}; \pi_t = \frac{\Pi_t}{A_{c,t}}; \hat{\pi}_t = \frac{\hat{\Pi}_t}{A_{c,t}}; \hat{k}_{x,t} = \frac{\hat{K}_{x,t}}{A_{c,t}}.
\end{align*}
\]
Note that consumption growth depends not only on technology specific to consumption good production, but also on efficiency of capital creation in this sector. Capital stock in consumption good sector is de-trended using $A_{c,t} A_{k,t}$:

$$k_{y,t} = \frac{K_{y,t}}{A_{c,t} A_{k,t}} = \frac{K_{y,t} A_{y,t}^{\frac{1}{1-\alpha_y}}}{A_{y,t} A_{k,t}^{\frac{1}{1-\alpha_y}}}.$$  

When consumption-good capital productivity $A_{k,t}$ is growing, the amount of consumption good spent on $K_{y,t}$ is growing slower than consumption along the balanced growth path. Also, consumption-sector capital rent measured in units of consumption good also needs to be de-trended: $r_{y,t} = R_{y,t} A_{k,t}$. Construction-sector capital is not affected by capital productivity process, so its rent $r_{x,t}$ does not require de-trending.

Let $A_{h,t}$ be the measure of total housing sector productivity. According to equation (1.19), housing sector employs consumption good and construction capital that originate from consumption good sector, along with labor that is subject to labor-augmenting productivity $A_{x,t}$. Then, the total housing sector productivity depends on both $A_{c,t}$ and $A_{x,t}$:

$$A_{h,t} = A_{c,t}^{\frac{\alpha_{xh} + \alpha_{xx}}{1-\alpha_{xh}}} A_{x,t}^{\frac{1-\alpha_{xh} - \alpha_{xx} - \alpha_{sh}}{1-\alpha_{xh}}}.$$  

Correspondingly, all the housing stock variables are de-trended using $A_{h,t}$.

$$h_t = \frac{H_t}{A_{h,t}}; \quad \hat{h}_t = \frac{\hat{H}_t}{A_{h,t}}; \quad \bar{h}_t = \frac{\bar{H}_t}{A_{h,t}}; \quad p_t = \frac{A_{h,t}}{A_{c,t}} P_t.$$  

Notice that the house price is de-trended using both consumption and housing sector productivity. That is, if the consumption good sector expands faster, the house price (measured in units of consumption goods per unit of housing stock) grows. In the end, all three technology processes ($A_{y,t}$, $A_{k,t}$, and $A_{x,t}$) affect the evolution of house price.
A.1.3 De-trended System

Savers

For savers, the de-trended versions of the budget constraint (1.7) and optimality conditions (1.8)–(1.11) are:

\[
\hat{c}_t + \hat{h}_t p_t + \hat{s}_t + \hat{k}_{x,t} = \frac{1 - \delta_h}{g_{h,t}} \hat{h}_{t-1} p_t \Omega_t + w_t + \hat{\pi}_t + \\
+ \frac{1 + r_{m,t}}{g_{c,t}} \hat{s}_{t-1} + \frac{r_{x,t} + 1 - \delta_x}{g_{c,t}} \hat{k}_{x,t-1} + \frac{1 + r_{y,t} - \delta_y}{g_{c,t} g_{k,t}} \hat{k}_{y,t-1};
\]  

(A.2)

\[
\frac{\nu_t}{\hat{c}_t} p_t = \frac{\nu_t \psi_t}{h_t} + \beta E_t \left[ \frac{\nu_{t+1} 1 - \delta_h}{\hat{c}_{t+1} g_{h,t+1}} p_{t+1} \Omega_{t+1} \right];
\]

(A.3)

\[
\frac{\nu_t}{\hat{c}_t} g_{c,t} = \beta E_t \left[ \frac{\nu_{t+1} r_{y,t+1} + 1 - \delta_y}{\hat{c}_{t+1} g_{c,t+1}} \right];
\]

(A.4)

\[
\frac{\nu_t}{\hat{c}_t} = \beta E_t \left[ \frac{\nu_{t+1} r_{x,t+1} + 1 - \delta_x}{\hat{c}_{t+1}} \right];
\]

(A.5)

\[
\frac{\nu_t}{\hat{c}_t} = \beta E_t \left[ \frac{\nu_{t+1} r_{m,t+1} + 1}{\hat{c}_{t+1} g_{c,t+1}} \right];
\]

(A.6)

Borrowers

For borrowers, the de-trended versions of the constraints (1.12)–(1.14) and optimality conditions (1.15)–(1.16) are:

\[
c_t + h_t p_t - b_t = \frac{1 - \delta_h}{g_{h,t}} h_{t-1} p_t \Omega_t (1 - \Gamma(\tilde{\omega}_t)) + w_t
\]

(A.7)

\[
1 + r_{m,t} = (1 + \tilde{r}_{m,t-1}) \frac{\Gamma(\tilde{\omega}_t) - \mu G(\tilde{\omega}_t)}{\tilde{\omega}_t}
\]

(A.8)

\[
\tilde{\omega}_t = \frac{b_{t-1}(1 + \tilde{r}_{m,t-1}) g_{h,t}}{h_{t-1} p_t \Omega_t (1 - \delta_h) g_{c,t}}
\]

(A.9)

\[
\beta E_t \left[ \frac{\nu_{t+1} c_t}{c_{t+1} g_{c,t+1} \nu_t} \Gamma'(\tilde{\omega}_{t+1}) \right] = E_t \left[ (\Gamma'(\tilde{\omega}_{t+1}) - \mu G'(\tilde{\omega}_{t+1}))/ (1 + r_{m,t+1}) \right]
\]

(A.10)

\[
\frac{\nu_t}{c_t} p_t = \frac{\nu_t \psi_t}{h_t} + \beta E_t \left[ \frac{\nu_{t+1} 1 - \delta_h}{c_{t+1} g_{h,t+1} p_{t+1} \Omega_{t+1}} (1 - \Gamma(\tilde{\omega}_{t+1})) \right] + \\
+ \frac{\nu_t}{c_t} E_t \left[ \frac{1 - \delta_h}{g_{h,t+1} p_{t+1} \Omega_{t+1}} (\Gamma(\tilde{\omega}_{t+1}) - \mu G(\tilde{\omega}_{t+1}))/ (1 + r_{m,t+1}) \right]
\]

(A.11)
Instead of the saver’s participation constraint, equation (A.8) defines the ex-post return on mortgage. The de-trended version of saver’s participation constraint (1.13) is defined collectively by (A.6), (A.8), and (A.9).

Production

Equations (1.17)–(1.25), and \( W_{x,t} = \bar{W}_{y,t} \) are combined and de-trended to get

\[
w_t = (1 - \alpha_y) n_{y,t}^{-\alpha_y} \left( \frac{k_{y,t-1}}{g_{c,t} g_{k,t}} \right)^{\alpha_y}; \tag{A.12}
\]

\[
r_{y,t} = \alpha_y n_{y,t}^{-1} \left( \frac{k_{y,t-1}}{g_{c,t} g_{k,t}} \right)^{\alpha_y-1}; \tag{A.13}
\]

\[
\Psi \hat{h}_t + (1 - \Psi) \hat{\pi}_t - \hat{h}_t = n_{x,t}^{1-\alpha_{xz}-\alpha_{xx}-\alpha_{xh}} \left( \frac{k_{x,t-1}}{g_{c,t}} \right)^{\alpha_{xk}} x_t^{\alpha_{xx}} \alpha_{xh}; \tag{A.14}
\]

\[
\pi_t = n_{x,t}^{1-\alpha_{zk}-\alpha_{xx}-\alpha_{xh}} \left( \frac{k_{x,t-1}}{g_{c,t}} \right)^{\alpha_{xk}} x_t^{\alpha_{xx}} \alpha_{xh} \frac{x_t}{n_{x,t}} - w_t n_{x,t} - r_{x,t} \frac{k_{x,t-1}}{g_{c,t}} - x_t; \tag{A.15}
\]

\[
p_t = \frac{x_t}{\alpha_{xx}} n_{x,t}^{1-\alpha_{zk}-\alpha_{xx}-\alpha_{xh}} \left( \frac{k_{x,t-1}}{g_{c,t}} \right)^{\alpha_{xk}} x_t^{\alpha_{xx}} \alpha_{xh}; \tag{A.16}
\]

\[
w_t = 1 - \alpha_{zk} - \alpha_{xx} - \alpha_{xh} \frac{x_t}{n_{x,t}}; \tag{A.17}
\]

\[
r_{x,t} = \frac{\alpha_{xk}}{\alpha_{xx}} \frac{x_t g_{c,t}}{k_{x,t-1}}; \tag{A.18}
\]

\[
\bar{h}_t = \frac{1 - \delta_{h}}{g_{h,t}} (\Psi h_t + (1 - \Psi) \hat{h}_t) \Omega_t. \tag{A.19}
\]

Market Clearing

Market-clearing conditions remain the same:

\[
\Psi b_t = (1 - \Psi) \hat{s}_t; \tag{A.20}
\]

\[
(1 - \Psi) \hat{k}_{y,t} = k_{y,t}; \tag{A.21}
\]

\[
(1 - \Psi) \hat{k}_{x,t} = k_{x,t}; \tag{A.22}
\]

\[
(1 - \Psi) \tilde{\pi}_t = \pi_t; \tag{A.23}
\]

\[
n_{x,t} + n_{y,t} = 1. \tag{A.24}
\]
Exogenous Processes

In the de-trended system, the levels of technology $A_{i,t}$ do not matter; only the growth rates $g_{i,t}$ are a part of state. To describe the three technologies ($i \in \{y, k, x\}$), equations (1.26) and (1.27) are expressed in terms of the growth rates:

\[
\ln g_{i,t} = \gamma_{i,t} + u_{i,t}; \quad (A.25)
\]
\[
\gamma_{i,t} = (1 - \rho_i) \gamma_i + \rho_i \gamma_{i,t-1} + v_{i,t}. \quad (A.26)
\]

Equations (1.28)–(1.30) are not affected:

\[
\ln \psi_t = \rho_\psi \ln \psi_{t-1} + (1 - \rho_\psi) \ln \psi + \epsilon_{\psi,t}; \quad (A.27)
\]
\[
\ln \Omega_t = \rho_\Omega \ln \Omega_{t-1} + \epsilon_{\Omega,t}; \quad (A.28)
\]
\[
\ln \nu_t = \rho_\nu \ln \nu_{t-1} + \epsilon_{\nu,t}. \quad (A.29)
\]

Equations (A.2)–(A.29) are a closed stationary system that can be approximated around the steady state. The steady state can be derived analytically; I omit the derivation of steady state and provide the log-linearized system below.

A.1.4 Log-linearized System

The system (A.2)–(A.29) is re-combined, simplified, and log-linearized around the steady state to obtain the following linear system, where variables with tilde stand for log-deviations from the steady state:
\[ \ddot{c}_t + k_y k_{y,t} + \ddot{k}_x k_{x,t} + \ddot{s}_t + \ddot{\bar{h}} + \ddot{h} p \left( 1 - \frac{1 - \delta_h}{g_h} \right) \bar{p}_t = w \ddot{w}_t + \hat{\bar{\pi}}_t \]

\[ + \frac{k_y}{\beta} (\ddot{k}_{y,t-1} - \ddot{g}_{k,t} - \ddot{g}_{c,t}) + \frac{k_y}{g_k g_c} r_y^r_{y,t} \bar{r}_{y,t} + \frac{k_y}{\beta} (\ddot{k}_{x,t-1} - \ddot{g}_{c,t}) + \frac{k_x}{g_c} r_x \bar{r}_{x,t} + \frac{\bar{s}}{g_c} r_m \bar{r}_{m,t} + \ddot{h} p \frac{1 - \delta_h}{g_h} (\ddot{h}_{t-1} + \Omega_t - \ddot{g}_{h,t}) \]  \hspace{1cm} \text{(A.30)}

\[ \beta \frac{1 - \delta_h}{g_h} E_t [\bar{p}_{t+1} + \bar{\Omega}_{t+1} - \ddot{g}_{h,t+1} - \ddot{c}_{t+1} + \bar{\nu}_{t+1}] = \bar{p}_t - \ddot{c}_t + \beta \frac{1 - \delta_h}{g_h} \bar{p}_t + \psi^c \left( \ddot{h}_t - \bar{\nu}_t \right) \]  \hspace{1cm} \text{(A.31)}

\[ \bar{\nu}_t - \ddot{c}_t = E_t [\bar{p}_{t+1} - \ddot{c}_{t+1} - \ddot{g}_{c,t+1} - \ddot{g}_{k,t+1} + \frac{r_y}{r_y + 1 - \delta_y} r_{y,t+1}] \]  \hspace{1cm} \text{(A.32)}

\[ \bar{v}_t - \ddot{c}_t = E_t [\bar{p}_{t+1} - \ddot{c}_{t+1} - \ddot{g}_{c,t+1} + \frac{r_x}{r_x + 1 - \delta_x} r_{x,t+1}] \]  \hspace{1cm} \text{(A.33)}

\[ \bar{v}_t - \ddot{c}_t = E_t [\bar{p}_{t+1} - \ddot{c}_{t+1} - \ddot{g}_{c,t+1} + \frac{r_m}{r_m + 1} r_{m,t+1}] \]  \hspace{1cm} \text{(A.34)}

\[ e \ddot{c}_t + h p \ddot{h}_t + h p \left( 1 - \frac{1 - \delta_h}{g_h} (1 - \Gamma) \right) \bar{p}_t - \ddot{b}_t = \]  \hspace{1cm} \text{(A.35)}

\[ = w \ddot{w}_t + h p \frac{1 - \delta_h}{g_h} ((1 - \Gamma)(\ddot{h}_{t-1} - \ddot{g}_{h,t} + \bar{\Omega}_t) - \Gamma' \bar{\omega} \bar{\nu}_t) \]  \hspace{1cm} \text{(A.36)}

\[ \ddot{p}_t = \psi^c \left( \ddot{w}_t + \ddot{c}_t - \ddot{h}_t \right) + \]  \hspace{1cm} \text{(A.37)}

\[ + \frac{1 - \delta_h}{g_h} (\ddot{h}_t (\Gamma - \mu G) + \beta (1 - \Gamma)) \left( E_t [\bar{p}_{t+1} + \bar{\nu}_{t+1} - \ddot{g}_{h,t+1} + \bar{\Omega}_{t+1}] - \bar{v}_t \right) + \beta \frac{1 - \delta_h}{g_h} (1 - \Gamma) (\ddot{c}_t - E_t [\ddot{c}_{t+1}]) \]  \hspace{1cm} \text{(A.38)}

\[ \ddot{\bar{\omega}}_t = g_{h,t} - \ddot{g}_{c,t} + \ddot{s}_{t-1} - \ddot{h}_{t-1} - \ddot{p}_t - \bar{\Omega}_t + \frac{r_m}{r_m + 1} \bar{r}_{m,t-1} \]  \hspace{1cm} \text{(A.39)}

\[ \bar{r}_{m,t} = \frac{r_m}{\bar{r}_m} \bar{r}_{c,t+1} + (1 + \bar{r}_m) \left( \Gamma' - \mu G' - \frac{r - \mu G}{\omega} \right) \ddot{\omega}_t \]

**Borrowers**

\[ e \ddot{c}_t + h p \ddot{h}_t + h p \left( 1 - \frac{1 - \delta_h}{g_h} (1 - \Gamma) \right) \bar{p}_t - \ddot{b}_t = \]  \hspace{1cm} \text{(A.35)}

\[ = w \ddot{w}_t + h p \frac{1 - \delta_h}{g_h} ((1 - \Gamma)(\ddot{h}_{t-1} - \ddot{g}_{h,t} + \bar{\Omega}_t) - \Gamma' \bar{\omega} \bar{\nu}_t) \]  \hspace{1cm} \text{(A.36)}

\[ \ddot{p}_t = \psi^c \left( \ddot{w}_t + \ddot{c}_t - \ddot{h}_t \right) + \]  \hspace{1cm} \text{(A.37)}

\[ + \frac{1 - \delta_h}{g_h} (\ddot{h}_t (\Gamma - \mu G) + \beta (1 - \Gamma)) \left( E_t [\bar{p}_{t+1} + \bar{\nu}_{t+1} - \ddot{g}_{h,t+1} + \bar{\Omega}_{t+1}] - \bar{v}_t \right) + \beta \frac{1 - \delta_h}{g_h} (1 - \Gamma) (\ddot{c}_t - E_t [\ddot{c}_{t+1}]) \]  \hspace{1cm} \text{(A.38)}

\[ \ddot{\bar{\omega}}_t = g_{h,t} - \ddot{g}_{c,t} + \ddot{s}_{t-1} - \ddot{h}_{t-1} - \ddot{p}_t - \bar{\Omega}_t + \frac{r_m}{r_m + 1} \bar{r}_{m,t-1} \]  \hspace{1cm} \text{(A.39)}

\[ \bar{r}_{m,t} = \frac{r_m}{\bar{r}_m} \bar{r}_{c,t+1} + (1 + \bar{r}_m) \left( \Gamma' - \mu G' - \frac{r - \mu G}{\omega} \right) \ddot{\omega}_t \]

**A.1.4.1 Production**

\[ \ddot{\bar{w}}_t = \alpha_y (\ddot{k}_{y,t-1} - \ddot{g}_{c,t} - \ddot{g}_{k,t} - \ddot{n}_{y,t}) \]  \hspace{1cm} \text{(A.40)}

\[ \ddot{\bar{r}}_{y,t} = (1 - \alpha_y) (\ddot{n}_{y,t} + \ddot{g}_{c,t} + \ddot{g}_{k,t} - \ddot{k}_{y,t-1}) \]  \hspace{1cm} \text{(A.41)}
\[ \Psi h \tilde{h}_t + (1 - \Psi)h \tilde{h}_t - \tilde{h}_t = \]
\[ = x \frac{1 - \alpha_x - \alpha_{xx} - \alpha_{kh}}{\alpha_{xx}} \tilde{n}_x,t + x z_{sk} (\tilde{k}_{x,t-1} - \tilde{g}_{c,t}) + x \tilde{x}_t + x z_{sh} \tilde{\tilde{h}}_t; \tag{A.42} \]
\[ \tilde{n}_t = \tilde{x}_t; \tag{A.43} \]
\[ \tilde{\gamma}_t = (1 - \alpha_{xx}) \tilde{x}_t - (1 - \alpha_{xk} - \alpha_{xx} - \alpha_{kh}) \tilde{n}_x,t - \alpha_{xk} (\tilde{k}_{x,t-1} - \tilde{g}_{c,t}) - \alpha_{xh}; \tag{A.44} \]
\[ \tilde{w}_t = \tilde{x}_t - \tilde{n}_x,t; \tag{A.45} \]
\[ \tilde{r}_{x,t} = \tilde{x}_t + \tilde{g}_{c,t} - \tilde{k}_{x,t-1}; \tag{A.46} \]
\[ \tilde{n}_{y,t} = - x_n \tilde{n}_x,t; \tag{A.47} \]
\[ \tilde{\tilde{h}}_t = \frac{\Psi h}{\Psi h + (1 - \Psi)h} \tilde{h}_{t-1} + \frac{(1 - \Psi)h}{\Psi h + (1 - \Psi)h} \tilde{\tilde{h}}_{t-1} + \tilde{\Omega} - \tilde{g}_{h,t}; \tag{A.48} \]

### A.1.4.2 Exogenous Processes

\[ \tilde{g}_{i,t} = \tilde{\gamma}_{i,t} + u_{i,t}, \quad i \in \{y, k, x\}; \tag{A.49} \]
\[ \tilde{\gamma}_{i,t} = \rho_{i} \tilde{\gamma}_{i,t-1} + v_{i,t}, \quad i \in \{y, k, x\}; \tag{A.50} \]
\[ \tilde{\psi}_t = \rho_{p} \tilde{\psi}_{t-1} + \epsilon_{\psi,t}; \tag{A.51} \]
\[ \tilde{\Omega}_t = \rho_{\Omega} \tilde{\Omega}_{t-1} + \epsilon_{\Omega,t}; \tag{A.52} \]
\[ \tilde{v}_t = \rho_{v} \tilde{v}_{t-1} + \epsilon_{v,t}. \tag{A.53} \]

### A.1.5 Steady-State Kalman Filter

Recall equations (1.34) and (1.35):

\[ \tilde{g}_{i,t} = \tilde{\gamma}_{i,t} + u_{i,t}; \]
\[ \tilde{\gamma}_{i,t} = \rho_{i} \tilde{\gamma}_{i,t-1} + v_{i,t}, \]
Under imperfect knowledge, \( \gamma_{i,t} \) is observable while \( \gamma_{i,t}, u_{i,t}, \) and \( v_{i,t} \) are not. Define \( \hat{\gamma}_{i,t|s} = E(\gamma_{i,t|s}|\tilde{g}_{i,0}, \ldots, \tilde{g}_{i,s}) \) as the forecasted value of the technological growth’s persistent component at time \( t \) given all available observations of the growth rate up to period \( s \); the forecast variance is \( \Sigma_{i,t|s} = E[(\gamma_{i,t} - \hat{\gamma}_{i,t|s})^2] \). Then, the following four equations summarize Kalman learning:

**Time update:**

\[
\hat{\gamma}_{i,t+1|t} = \rho_i \hat{\gamma}_{i,t|t}; \\
\Sigma_{i,t+1|t} = \rho_i^2 \Sigma_{i,t|t} + \sigma_{i,v}^2;
\]

**Measurement update:**

\[
\tilde{\gamma}_{i,t|t} = \hat{\gamma}_{i,t|t-1} + \Sigma_{i,t|t-1}(\Sigma_{i,t|t-1} + \sigma_{i,u}^2)^{-1}(\tilde{g}_t - \hat{\gamma}_{i,t|t-1}); \\
\Sigma_{i,t|t} = \Sigma_{i,t|t-1} - \Sigma_{i,t|t-1}(\Sigma_{i,t|t-1} + \sigma_{i,u}^2)^{-1} \Sigma_{i,t|t-1};
\]

This system can be combined into two equations:

\[
\begin{align*}
\hat{\gamma}_{i,t|t} &= \hat{\gamma}_{i,t-1|t-1} + \Sigma_{i,t|t-1}(\Sigma_{i,t|t-1} + \sigma_{i,u}^2)^{-1}(\tilde{g}_t - \hat{\gamma}_{i,t-1|t-1}); \\
\Sigma_{i,t+1|t} &= \rho_i^2 \Sigma_{i,t|t-1} - \rho_i^2 \Sigma_{i,t|t-1}(\Sigma_{i,t|t-1} + \sigma_{i,u}^2)^{-1} + \sigma_{i,v}^2,
\end{align*}
\]

Equation (A.55) is the Ricatti recursion, and the steady-state filtering implies that the forecast variance \( \Sigma_{i,t+1|t} \) is constant. Solving (A.55) for the steady-state variance \( \Sigma_i \) yields the expression for the steady-state Kalman gain:

\[
\lambda_i = \frac{d_i - (1 - \rho_i^2) + \sqrt{(1 - \rho_i^2)^2 + d_i^2 + 2(1 + \rho_i^2)d_i}}{2 + d_i - (1 - \rho_i^2) + \sqrt{(1 - \rho_i^2)^2 + d_i^2 + 2(1 + \rho_i^2)d_i}},
\]
where $d_i = \sigma_{\nu,i}^2/\sigma_{\mu,i}^2$. Letting $\hat{\gamma}_{i,t} = \hat{\gamma}_{i,t|t}$, the updating equation for the value of persistent component becomes

$$\hat{\gamma}_{i,t} = \lambda_t \hat{g}_{i,t} + (1 - \lambda_t) \rho_t \hat{\gamma}_{i,t-1}.$$ 

### A.1.6 Decomposition of Technology Processes

Recall equations (1.26) and (1.27):

for $i \in \{y, x, k\}$, \( \ln A_{i,t} = \ln A_{i,t-1} + \gamma_{i,t} + u_{i,t} \), \( u_{i,t} \sim N(0, \sigma_{\nu,i}^2) \), i.i.d.

$$\gamma_{i,t} = (1 - \rho_i) \gamma_{i} + \rho_i \gamma_{i,t-1} + v_{i,t}, \quad v_{i,t} \sim N(0, \sigma_{\nu,i}^2), \text{ i.i.d.}$$

Define $\tilde{\gamma}_{i,t} = \gamma_{i,t} - \gamma_i$ and reformulate the technology process:

$$\ln A_{i,t} = a_{i,t} + \gamma_{i,t} - \rho_i \tilde{\gamma}_{i,t};$$

$$a_{i,t} = a_{i,t-1} + \frac{1}{1-\rho_i} v_{i,t} + u_{i,t}.$$ 

The last equation illustrates how the technology process is captured by different parts of the system (1.31): $a_{i,t}$ is the non-stationary stochastic component that is captured by the state vector $s_{2,t} = (a_{y,t}, a_{k,t}, a_{x,t})'$ defined by equation (1.33); and $\gamma_{i,t}$ is the deterministic trend component captured by the term $\Phi_3 t$ in the system. The last component $\tilde{\gamma}_{i,t}$ of the technology process is stationary and is a part of vector $s_{1,t}$. Notice that the linear system (1.33) describing the evolution of $s_{2,t}$ is the exact solution—that is, log-linearization is applied only to the stationary part of the model, and the fact that the model is nonstationary does not add imprecision to log-linearization as a method to approximate the solution of the model.

### A.2 Description of the Data

**Aggregate Consumption**: real personal consumption expenditures, nondurable goods and services (seasonally adjusted, chained 2009 dollars, Table 2.3.3), divided by civil-
ian noninstitutional population (CNP16OV), and logged. Source: Bureau of Economic Analysis (BEA), Bureau of Labor Statistics (BLS)

**Nonresidential Investment**: real private fixed nonresidential investment (seasonally adjusted, chained 2009 dollars, Table 5.3.6), divided by civilian noninstitutional population (CNP16OV), and logged. Source: BEA, BLS

**Residential Investment**: real private fixed residential investment (seasonally adjusted, chained 2009 dollars, Table 5.3.6), divided by civilian noninstitutional population (CNP16OV), and logged. Source: BEA, BLS

**House Price**: All-Transactions House Price Index (USSTHPI), divided by Consumer Price Index for All Urban Consumers, all items less shelter (CUUR0000SA0L2), logged, and cleared of seasonal component using X-12-ARIMA algorithm. Source: Federal Housing Finance Agency, BLS

### A.3 Redistributional impact of uncertainty

Figure A.1 shows the redistributional impact of imperfect knowledge in case of a negative persistent shock to consumption technology. Initially, households over-optimistically bet on transitory shock and quick recovery in the near future; as a result, the house price and the scope of mortgage lending remain higher. Eventually, households learn that the recession is long. The result is lower house price and higher default rate on mortgages that prove to be oversized ex post. Borrowers lose net worth and experience a fall in consumption and housing purchases. Savers, on the contrary, are able to buy cheap housing and maintain consumption. In effect, the ex-post excessive optimism redistributes wealth from borrowers to savers. The scope of redistribution is larger if the persistent shock is less volatile, so that households do not believe in a long recession and learn about it slowly.
Figure A.1: Impulse responses: redistributional effect of uncertainty in case of persistent shock to consumption technology $v_{y,t}$. Percentage deviations of non-detrended variables from the balanced growth path due to one-standard-deviation negative shock. Solid lines represent the case of certainty; dashed lines represent uncertainty. For default rate, the values are absolute, compared with their steady-state values (dotted lines). Numbers near the arrows show where the deviations will stabilize after 400 quarters.

Appendix B

SUPPLEMENT TO CHAPTER 2

In this section of the appendix, figures B.1–B.6 represent the quality of the output simulated by the MH algorithm (plotted series, sample auto-correlations, and recursive means). Figures B.7–B.8 plot historical variance decomposition of the observed data as interpreted by the model with perfect knowledge about persistence of shocks.
Figure B.1: Simulated series: model with perfect knowledge. MH algorithm is used to obtain 450,000 simulations, where the first 49,999 are counted as burn-in stage. Then, every 100th observation is retained to get the sample of size $M = 4,001$. 
Figure B.2: Simulated series: model with imperfect knowledge. MH algorithm is used to obtain 450,000 simulations, where the first 49,999 are counted as burn-in stage. Then, every 100th observation is retained to get the sample of size $M = 4,001$. 
Figure B.3: Serial correlation of the series: model with perfect knowledge. For the shock processes that have near-zero standard deviations, the simulated series of the standard deviations and the associated AR coefficients exhibit high serial correlation.
Figure B.4: Serial correlation of the series: model with imperfect knowledge. For the shock processes that have near-zero standard deviations, the simulated series of the standard deviations and the associated AR coefficients exhibit high serial correlation.
Figure B.5: Recursive sample means: model with perfect knowledge. Series that exhibit high serial correlation converge poorly.
Figure B.6: Recursive sample means: model with imperfect knowledge. Series that exhibit high serial correlation converge poorly.
Figure B.7: Historical variance decomposition, perfect knowledge, aggregate consumption and capital investment. White dots represent the observed percentage growth in the variables; bar heights represent contributions of each of the shock to the growth.
Figure B.8: Historical variance decomposition, perfect knowledge, residential investment and house price. White dots represent the observed percentage growth in the variables; bar heights represent contributions of each of the shock to the growth.
Bibliography


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